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THE DOCTRINE OF GERMS,

OR

THE INTEGRATION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS WHICH OCCUR IN MATHEMATICAL PHYSICS.

BY

Samuel

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PREFACE.

THE method of integration by means of Germs adopted in this Treatise is based on the admitted principle that in the work of integrating a proposed differential equation we are free to avail ourselves of the advantages offered by any distinctive peculiarities that are perceived to exist in the equation itself prior to its integration. Any such peculiarity will, as a matter of course, impress a corresponding peculiarity on the integral to be found. Equations will therefore be classified according to their distinctive peculiarities, those peculiarities being indicated by the particular ways in which germs may be connected with the variables of a differential equation without disturbing or in any way affecting its form.

In this Treatise the differential equations that will be brought before the reader are all linear and partial, in consequence of which the doctrine of germs suitable for such equations admits of being presented in a form that is easily reduced to a system of singular efficiency. But there are certain other equations that are not linear, and which therefore do not fall under the system that will be developed in the following pages. The equation $\left(\frac{du}{dx}\right)^2 \frac{d^2u}{dt^2} = \frac{d^2u}{dx^2}$ is one of this kind, for its form will not be affected if ax be written for x; neither will it be affected if bu and bt be written simultaneously for u and t; and these arbitrary constants a, b, will therefore necessarily find a place (either explicitly or implicitly) in its integral. Also as only x takes the constant a, this indicates a peculiarity of x as to the manner in which it can appear in the integral. That u and t are alike related to b is indicative of the existence of a peculiar relation of u to t. And, further, we may write $u+A$ for u, $x+g$ for x, and $t+h$ for t (the constants A, g, h being perfectly arbitrary) without affecting the form of the equation. This shews that if U be an integral of it, so likewise will be the following,

$$\epsilon^{\frac{gd}{dx}+\frac{hd}{dt}}(U+A).$$

We may therefore presume that as there has been found for equations that are linear a "*Doctrine of Germs*," so there may be a possible "*Doctrine of Germ-like Constants*" for equations that are not linear.

In Chap. III. is introduced a theory of "Symbolical Equivalences." The subject is regarded from a point of view which may be considered as in some degree new. The exigencies of this Essay did not seem likely to require the complete development of this Theory; and in consequence of this only so much detail is given as was likely to be wanted in subsequent chapters. The Theory is capable of throwing light on several troublesome known paradoxes which have often been a source of perplexity to the Mathematical Student.

As the author was induced to undertake the development of the Doctrine of Germs by a desire to accomplish the complete integration of Laplace's Equation, and the consequent discovery of the general form of Laplace's Functions, he has deemed it to be of some possible advantage to obtain and to exhibit those results in several forms and aspects; hoping also that by this diversity the reader would be the more disposed to accept results which so confirm one another.

It has been taken as an admitted definition of Laplace's Functions, that any quantity which satisfies the equation (1) of

Art. 127, is a Laplace's Function. Laplace's Equation is usually given in two forms, viz. those in Arts. 124, 127; and to both of these results have been adapted. The first of these forms does not contain r; the second does; and it will be seen that r enters the latter in a manner that demands peculiar management, and when so managed leads to remarkable results, which can hardly fail to throw some light on the usually received theory of Laplace's Functions.

There appears to be an exceptional case of these functions which the reader will find in Art. 131.

It is needful to advise the reader before he enters upon the task of reading some parts of this Treatise, that when arbitrary constants occur in a general integral of a linear equation it will consist of the sum of several subgeneral integrals; and each of these constants being arbitrary by nature will retain their arbitrary character when multiplied by a definite numerical quantity; and if such a constant be separated into several arbitrary parts, each part will be as arbitrary as the original constant. Hence in altering the form of a general integral it will not be necessary to preserve the identity of any such constants, but only the quality of their independent arbitrariness; and thus we need not observe with respect to them the usual requirements of algebraic rules of reduction in passing from step to step. The arbitrary constants referred to are used merely to indicate the absolute independence of the subintegrals or subgeneral integrals of which they are the respective coefficients. (See Art. 42 for an example of what is here alluded to.)

The author has pleasure in acknowledging the valuable assistance rendered him by Mr Greenhill, M.A., Fellow of Emmanuel College, in supervising this Essay in its passage through the press.

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CORRIGENDA.

Page 22, line 6; *for* subsequent *read* subgeneral.
,, 25, line 8; *for* $+c'z)$ *read* $+c'z)\,u$.
,, 29, line 8; *for* algbraic *read* algebraic.
,, 46, line 11; *for* ϵ^{Mt^2} *read* ϵ^{M^2t}.

THE INTEGRATION

OF

LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

CHAPTER I.

INTRODUCTORY REMARKS.

IN the present work we have not to deal with Linear Partial Differential Equations in general, but only with such as are known to be of difficult integration, and which have been found to present themselves in connexion with the application of Mathematics to various branches of Natural Philosophy.

This task we have undertaken not as a branch of analytical enterprise, but as a contribution to the resources of those philosophers who think it a matter of importance that Physical Theories should be subject to the severe test of mathematical confirmation. And in the execution of this task we believe that we shall have the privilege of developing a method of integration which may be regarded as new, and that is singularly well adapted to the integration of certain equations which have been found intractable by ordinary methods.

1. In some cases a remarkable degree of uncertainty and intricacy besets the answer to the question,—when may an integral of a linear differential equation be rightly styled *its general integral?* The following are some of the reasons of this uncertainty.

If one of the independent variables (as x) of a linear differential equation occur therein not as a symbol of quantity but only as a symbol of differential operation $\left(\text{as } \frac{d}{dx}\right)$, then (supposing U an integral of the equation) not only will U be an integral but so likewise will each one of the quantities

$$A\frac{dU}{dx},\quad B\frac{d^2U}{dx^2},\quad C\frac{d^3U}{dx^3},\ \ldots \text{ ad infinitum},$$

and (speaking generally) we thus can out of a single known integral create an unlimited number of new integrals all different from the original and from each other.

And not only can we thus create new integrals, but out of these new ones we can by addition of all of them, or of a few of them selected arbitrarily, create an unlimited number of fresh integrals different from U and from each of those previously found from U by differentiation.

Now as we have the right of forming known integrals into groups as we please, and from the nature of the integrals of a linear equation each group will be a new integral, we are obliged to come to the conclusion that it is not easy to see on what principle we can say of some one of the infinite mass of integrals a proposed linear differential may have, that it is the general integral.

2. We shall be able to shew, in the case of every linear differential equation of the class supposed in the preceding article, that it admits of integrals of a peculiar kind, forming a distinct class, and they are generally infinite in number, and when added together form a sum which is equivalent to the general integral.

For distinctness of reference we shall denominate these integrals *subintegrals;* and when we speak of those collectively which belong to the same differential equation we shall denominate them a *family* of subintegrals.

If then P, Q, R, S,... be the individual members of a family of subintegrals, and u be the general integral of the same equation as that to which they all individually belong, we shall represent the relation between u and $P, Q, R, \ldots$ by the following equation,

$$u = AP + BQ + CR + DS + \ldots \text{ ad infin.}\ldots\ldots\ldots(1),$$

$A, B, C, D, \ldots$ being arbitrary constants, the use of which in this equation is, to indicate the absolute independence of $P, Q, R, \ldots$ as integrals of the proposed equation.

3. Sometimes the general integral (1) will divide itself by some peculiarity of form into two or more distinct parts; and to these independent parts we intend to refer under the designation of *subgeneral integrals.*

The number of such subgeneral integrals that belong to a proposed differential equation is generally dependent on the order or some other peculiarity of the equation.

4. When we know a family of subintegrals (as $P, Q, R, \ldots$) we can by grouping them into different heaps, and finding the sum of each group, take the various sums thus formed as a family of subintegrals; and as a family it will be symbolically equivalent to the original family ($P, Q, R, \ldots$), though the members of the new family may happen to have no similitude of form to the members of the other.

Their equivalence results from the one fact that each of them explicitly or implicitly contains all the members of the subintegral family of which u (the general integral) is known to be constituted.

Thus a family of subintegrals always admits of being recast by grouping, by summation of series, and other means whereby a change of the forms of its members is effected.

This is important because it brings forward the question,—what will be the most convenient forms in which a family of subintegrals can be obtained?

One answer to this would be;—let the members of the family be cast in such forms, that if any one member of the family can be found all the other members may be obtained from it by simple and repeated differentiations or integrations of that one member with respect to one or more of the independent variables contained in the proposed differential equation.

We shall in due time shew how and under what conditions this may be done.

5. As the existence of subintegrals is but little known, we shall here add the following illustration.

Let $\frac{d^2u}{dx^2} = \frac{du}{dy}$ be a differential equation; then we can at a glance detect the following as independent integrals of it:

$$1;\ \frac{x}{1};\ \frac{x^2}{1.2} + y;\ \frac{x^3}{1.2.3} + \frac{xy}{1.1};\ \frac{x^4}{1.2.3.4} + \frac{x^2y}{1.2.1} + \frac{y^2}{1.2};\ \ldots\ldots$$

each of which can be obtained from the one before it by integration with regard to x, and correction with regard to y. If this be the whole family then

$$u = A + B\frac{x}{1} + C\left(\frac{x^2}{1.2} + \frac{y}{1}\right) + D\left(\frac{x^3}{1.2.3} + \frac{xy}{1.2}\right) + \ldots \text{ ad infin.},$$

A, B, C, ... being arbitrary and absolutely independent constants.

6. We shall see in future articles that changes of the independent variables of a proposed differential equation can be sometimes made without producing any effect whatever on the *form* of the equation.

Whenever this can be done, the same changes of the same variables may be made in any known integral of the proposed equation without depriving that integral (however much changed in form thereby) of its property of being still an integral, or of its generality as an integral.

Also if the known integral should happen to be a particular and not a general integral the change of variables just described would introduce such a change of form of the integral itself as might bring it nearer to the form of a general integral by introducing new arbitrary constants which we should be at liberty to treat as germs not existing in the original particular integral.

7. We have just used the word *germ;* let us now explain what we mean by it.

We are aware that integration generally introduces to our notice in the integral certain constant quantities which have no existence in the differential equation itself. Such constants are in fact the offspring of integration; and are generally denominated *arbitrary constants.* The use of such constants in problems is well known.

This designation however is not sufficient for our purpose, and we intend to speak of them, under certain circumstances, as *germs,* or *germ-constants.* For as each of the variables of a proposed linear differential equation is constant with reference to every operating differential symbol contained therein except its own, so an arbitrary constant (*germ*) is constant with reference to all the differential symbols except its own; and it or any function of it contained in u may be operated on by its own symbol of differential operation, though no such symbol is contained in the proposed equation.

We may therefore consider a germ as being a *new* independent variable, i.e., an independent variable that is not contained in the differential equation itself, but only in its integral.

Thus ϵ^{ax+a^2y} is an integral of the equation $\frac{d^2u}{dx^2}=\frac{du}{dy}$; but the constant a is constant only in reference to $\frac{d}{dx}$ and $\frac{d}{dy}$, but not in reference to $\frac{d}{da}$; and therefore in this integral we may consider a either an arbitrary constant, or a new independent variable additional to x and y; and this is the property to which we refer when we call a a *germ.*

8. We shall find it convenient to be able to speak of certain germs under specific names, which will refer to the manner in which a germ in an integral may happen to stand (actually or virtually) connected with its independent variable. Thus if a germ g and an independent variable x stand connected in an integral by addition (as distinguished from multiplication), (as in the form $x \pm g$) we shall refer to g as the *minor germ of x.*

But if the connexion be of the nature of multiplication $\left(\text{as } gx \text{ or } \frac{x}{g}\right)$ we shall speak of g as the *major germ of x.*

9. Germs may also be regarded as being *general*, or *real.*

A *general germ* may receive any value whether real or imaginary.

A *real germ* may receive only such values as are not imaginary.

Nevertheless a general germ may be perfectly represented by means of two real germs. Thus if K be a general germ, the equation

$$K = M + im,$$

in which M and m are independent real germs, will perfectly represent K. Hence a general germ is equivalent symbolically to two real germs.

We shall throughout this Treatise use the two quantities i and j in the ambiguous senses implied by the two independent equations following,

$$i^2 = -1, \text{ and } j^2 = +1;$$

and both i and j will be regarded as independently carrying with them their proper double algebraic signs.

10. There are functions of x which cannot be expanded by MacLaurin's Theorem; and therefore the series $A + Bx + Cx^2 + \dots$, in which the coefficients $A, B, C, \dots$ are all arbitrary, does not symbolically represent a perfectly arbitrary function of x; but $Ax^\alpha + Bx^\beta + Cx^\gamma + \dots$ in which $A, B, C, \dots$ are arbitrary and $\alpha, \beta, \gamma, \dots$ not limited by the condition that they are to be positive integers, symbolically represents a perfectly arbitrary function.

We may distinguish these cases, when necessary, by denominating the former a MacLaurin's arbitrary function.

11. To find the potentiality of a germ when it occurs in an integral only as an index or power of a function of the independent variables.

Let $W\omega^m$ be an integral of a proposed equation, W and ω being functions of the independent variables, and m being a germ that occurs only as the index of the quantity denoted by ω.

We may give to m an infinite series of different values $\alpha, \beta, \gamma, \ldots$ at pleasure; and each one of these values of m will furnish us with an independent integral; and all the integrals so obtained we may unite in a single integral in the following manner,

$$W\omega^m = W(A\omega^\alpha + B\omega^\beta + C\omega^\gamma + \ldots \text{ ad infin.}),$$

$A, B, C, \ldots$ being independent arbitrary constants.

But $\alpha, \beta, \gamma, \ldots$ being arbitrary also, the series

$$A\omega^\alpha + B\omega^\beta + C\omega^\gamma + \ldots$$

will represent an arbitrary function of ω of the most perfectly arbitrary kind;

$$\therefore\ W\omega^m = WF(\omega).$$

Hence a germ when it occurs in an integral as an index only is potentially equivalent to an arbitrary function of the most general kind.

The converse is manifestly true, viz., if $F(\omega)$ be a perfectly general arbitrary function of ω, then will $F(\omega) = \omega^m$ symbolically.

But if $F(\omega)$ represent a MacLaurin's series only, then it does not follow that $F(\omega) = \omega^m$ symbolically, for in this case $F(\omega)$ is clogged with the condition that the powers of ω in the expansion of $F(\omega)$ must be positive integers. For such a function we shall therefore when it occurs have to find a potential equivalent clogged with the same condition. ω^m is, as we have said, too general, and may consequently (if incautiously used) lead us into error when we come to the generalizing of results obtained.

12. It will be a convenience to be allowed sometimes to represent the product $1 . 2 . 3 \ldots n$ by the symbol $n!$.

CHAPTER II.

SOME GENERAL PROPERTIES OF GERMS.

13. LET $\varpi . u = 0$ represent a general linear partial differential equation of any number of independent variables, u being the dependent variable, and ϖ denoting the compound operating symbol. Also let U denote any integral of this equation containing a germ. Denote the germ by c, and expand U in a series according to the powers of c;

$$\therefore \quad U = Pc^p + Qc^q + Rc^r + \dots\dots\dots\dots\dots\dots (1),$$

in which p, q, r,... are definite indices, and P, Q, R,... are functions of the independent variables.

Operate on each member of this equation with ϖ, noting that $\varpi (U) = 0$;

$$\therefore \quad 0 = (\varpi . P)\, c^p + (\varpi . Q)\, c^q + (\varpi . R)\, c^r + \dots.$$

Now c being a germ is an arbitrary independent variable, and consequently this must be an identical equation;

$$\therefore \quad 0 = \varpi . P, \quad 0 = \varpi . Q, \quad 0 = \varpi . R, \dots,$$

that is, P, Q, R,... are independent integrals of the proposed equation; they are, in fact, the family of subintegrals, the members of which are rendered independent by the fact that the powers of c in equation (1) are all different. They owe their independence to the presence of a germ in U.

But we can preserve their independence another way, and at the same time unite the subintegrals in a single integral, thus

$$U = AP + BQ + CR + \dots\dots\dots\dots\dots\dots (2),$$

in which A, B, C, ... are arbitrary constants.

Comparing this with (1) we perceive that the different powers of a germ in an expanded integral are symbolically equivalent to independent arbitrary constants.

We have now obtained the power of eliminating a germ by expansion of an integral according to the powers of that germ.

And, conversely, we can eliminate arbitrary constants, which belong to a series each term of which is a subintegral, by means of an extemporized germ.

14. If the integral U in the preceding article should happen to contain a second independent germ (n), then as only m has been eliminated the subintegrals P, Q, R, ... will each contain the germ n constituting each of them a germ-integral of the proposed equation. From each of these n may therefore be independently eliminated, and each of them will be thereby resolved into its own constituent subintegrals. Thus we have before us the fact that a subintegral may be also a germ-integral, and in this character resolvable into subintegrals of a more elementary class: and the ultimate subintegrals are those which do not contain a germ, and into which a germ cannot be introduced.

We may arrive by one step at the ultimate subintegrals of U by expanding U in the first instance in a series according to the powers of both the germs m and n, and then writing independent arbitrary constants for the germs and their powers and different combinations.

The converse is also true, viz. that we may eliminate arbitrary constants by means of the powers and the combinations of the powers of two or more independent germs.

Thus if

$$U = A + (Bx + B'y) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

we may (if A, B, C, ... be independent arbitrary constants) express a symbolical equivalent to this series by means of two independent germs m, n, thus,

$$U = A\left\{1 + (mx + ny) + \left(m^2\frac{x^2}{1.2} + mn\frac{xy}{1.1} + n^2\frac{y^2}{1.2}\right) + \dots\right\}$$
$$= A\epsilon^{mx+ny}.$$

We may reverse this process and at one step assume, if m, n be independent germs, the following symbolical equivalence,

$$\epsilon^{mx+ny} = A + (Bx + B'y) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

15. As a matter of experience we know that different physical and geometrical problems lead us to the same forms of partial differential equations. The equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

is a well-known example of this. Hence each member of a family of subintegrals being a complete integral in itself of its kind, expresses the solution of a particular problem in physics or geometry, which can exist independently; and it also expresses what may be called an independent elementary state of matter or some affection thereof; or some imaginable independent geometrical condition of an elementary nature. The subintegrals being independent, represent properties which can exist independently in nature.

Hence whenever an integral can be resolved into elementary subintegrals we have in such cases this fact before us; that the problem which brought us to the corresponding differential equation is really of a compound nature and capable of being resolved into a number of elementary problems, the superposition of which in their proper proportions is equivalent to the original problem.

Hence, also, one problem may require for its complete representation one set (or group) of members selected from the whole family of subintegrals; and another problem another set of members. And thus comes into use that important property of all linear differential equations, that the sum of any of the members of a subintegral family is an integral of the same differential equation and represents the superimposed action of so many different geometrical or physical properties.

The selection of the group of individual members of a family of subintegrals suitable for the solution of a given problem is sometimes a work of difficulty, and will always make a demand upon the investigator's ingenuity.

16. The connexion between the general integral of an equation and its family of subintegrals is exhibited in Art. 13, in the equation

$$U = AP + BQ + CR + \dots$$

Now as these coefficients A, B, $C, \dots$ are arbitrary and independent it would appear on the face of this series that it is incapable of being expressed in a finite form, there being no law connecting the coefficients with one another. We have seen however that by means of a germ and its powers we can symbolically represent simultaneously in a finite form both the integral itself and the arbitrariness and the independence of the coefficients. There are cases in which this can be accomplished in more forms than one.

It is always an important object to assume a germ and its powers in such forms and with such a law of coefficients as may enable us to sum the series which is constituted of subintegrals in a finite form. This cannot always be done; and when it cannot, then the substitution of a germ and its powers for A, B, $C, \dots$ is generally useless; and recourse must be had to the use of other means.

To make our meaning in this matter quite clear, we will produce an example (of a very simple kind) of the process of gathering up a whole family of subintegrals into one finite form which contains them all, and yet at the same time implicitly preserves the individuality and the independence of every member of the family. This is the chiefest of all the properties of a germ, and renders the doctrine of germs of much importance. Nothing can well exceed their utility in the discovery of symbolical equivalences; and in the transformation of integrals by means of these equivalences.

17. Let it be granted that the following quantities constitute as a whole a complete family of subintegrals, belonging to a certain linear differential equation:

$$1;\ \frac{x}{1};\ \frac{x^2}{1.2}+\frac{y}{1};\ \frac{x^3}{3!}+\frac{xy}{1.1};\ \frac{x^4}{4!}+\frac{x^2y}{1.2.1}+\frac{y^2}{1.2};\ \&c.$$

We combine them on the principle of superposition into a single integral U by means of arbitrary constants thus,

$$U=A+B\frac{x}{1}+C\left(\frac{x^2}{1.2}+\frac{y}{1}\right)+D\left(\frac{x^3}{3!}+\frac{xy}{1.1}\right)$$
$$+E\left(\frac{x^4}{4!}+\frac{x^2y}{1.2.1}+\frac{y^2}{1.2}\right)+\ldots$$

in which $A, B, C, \ldots$ are absolutely arbitrary and independent.

We now assume an extemporised germ m, and use it and its powers to replace (or eliminate) the arbitrary constants $A, B, C, \ldots$

$$\therefore\ U=A\left\{1+m\frac{x}{1}+m^2\left(\frac{x^2}{1.2}+\frac{y}{1}\right)+m^3\left(\frac{x^3}{3!}+\frac{xy}{1.1}\right)+\ldots\right\}$$
$$=A\left(1+\frac{mx}{1}+\frac{m^2x^2}{1.2}+\ldots.\right)\left(1+\frac{m^2y}{1}+\frac{m^4y^2}{1.2}+\ldots.\right)$$
$$=A\epsilon^{mx+m^2y}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

We have here prefixed the common coefficient A, because every integral of a linear equation takes an arbitrary general coefficient. In this simple form of the complete integral are contained (without loss of their individual independence) all the members of the subintegral family because m is a germ.

We may also now obtain the differential equation of which the above are the independent subintegrals. For by differentiating equation (1) with respect to its independent variables x, y, we find

$$\frac{d^2U}{dx^2}=\frac{dU}{dy}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Now in reference to the form of this differential equation we may remark; that it would not be changed or in any way affected were we to write $x+g$ and $y+h$ for x and y; neither would its form be changed by writing mx and m^2y for x and y. We shall express this by saying that this equation allows its independent variables to take both major and minor germs.

In the above integral (1) the major germ m enters explicitly; but if we write $x+g$, $y+h$ for x, y in it, it takes the following form,

$$A\epsilon^{m(x+g)+m^2(y+h)} = A\epsilon^{mg+m^2h}\cdot\epsilon^{mx+m^2y} = A\epsilon^{mx+m^2y}.$$

Hence the exponential form of the integral (1) can be in no way affected in generality by the introduction of major and minor germs, the former being already present in it explicitly; and the latter implicitly, inasmuch as they may be said to lie hidden in the external arbitrary coefficient A.

18. If one of the independent variables of a proposed linear differential equation can take a minor germ, the family of subintegrals can by means of that germ be cast in such a form that all the subintegrals can be obtained from any one of the family by simple integration and differentiation with respect to that variable. (See Art. 4.)

Let $\varpi . u = 0$ be a linear differential equation which allows one of its independent variables (as x) to take a minor germ g.

Then as the writing of $x+g$ for x in $\varpi . u = 0$ produces no change, we may do the same in any integral of $\varpi . u = 0$ without destroying it as an integral; and as the substitution of $x+g$ for x in an integral would introduce the new germ g the generality of the integral would not be diminished; but on the contrary it would be increased, unless the integral in which the substitution is made be itself perfectly general. Hence the general integral must of necessity be of such a form that the substitution of $x+g$ for x in it cannot affect its perfect generality,

$$\therefore\ u = F(x+g, y, z, \dots).$$

Let this be expanded by Taylor's Theorem in powers of g;

$$\therefore\ u = \left(1 + \frac{g}{1}\cdot\frac{d}{dx} + \frac{g^2}{2!}\cdot\frac{d^2}{dx^2} + \frac{g^3}{3!}\cdot\frac{d^3}{dx^3} + \dots\right) F(x, y, z, \dots)$$

$$= \left(A + B\frac{d}{dx} + C\frac{d^2}{dx^2} + D\frac{d^3}{dx^3} + \dots\right) F(x, y, z, \dots).$$

The last step is by Art. 13; and A, B, C, ... are arbitrary constants.

Hence A, B, C, ... are the coefficients of the subintegrals, which therefore are in their order as follows [we denominate $F(x, y, \ldots)$ the *first* subintegral],

$$F(x, y, \ldots);\ \frac{d}{dx}F(x, y, \ldots);\ \frac{d^2}{dx^2}F(x, y, \ldots);\ \&c.$$

If any one of this family become known, then the whole family may be found from that one, by integration and by differentiation with regard to x.

From this property of minor germs, it becomes a matter of no little importance, whenever it can be done, to reduce a proposed equation which does not allow any of its independent variables to take a minor germ to a form that will allow a minor germ.

19. The potentiality of a minor germ may always be represented in an equivalent form by means of an arbitrary function; that function being, however, not one of quantitative symbols but of symbols of operation.

For when one of the independent variables (as x) takes a minor germ (as g) we have seen that

$$\begin{aligned} u &= F(x+g, y, \ldots) \\ &= \left(1 + \frac{g}{1}\frac{d}{dx} + \frac{g^2}{1.2}\frac{d^2}{dx^2} + \ldots\right) F(x, y, \ldots) \\ &= \left(A + B\frac{d}{dx} + C\frac{d^2}{dx^2} + \ldots\right) F(x, y, \ldots) \\ &= \phi\left(\frac{d}{dx}\right) . F(x, y, \ldots). \end{aligned}$$

If another variable (as y) take an independent minor germ h, we should find in the same way that

$$u = \phi\left(\frac{d}{dx}, \frac{d}{dy}\right) . F(x, y, \ldots),$$

and so on to any number of variables taking independent minor germs.

20. If an independent variable (as x) takes a minor germ g, the general integral of the equation will always admit of perfect symbolical expression in the form of an infinite series according to positive integer powers of that variable.

For in this case

$$\begin{aligned} u &= F(x+g, y, \ldots) \\ &= F(g+x, y, \ldots) \\ &= \left(1 + \frac{x}{1}\frac{d}{dg} + \frac{x^2}{1.2}\frac{d^2}{dg^2} + \ldots\right) F(g, y, \ldots), \end{aligned}$$

which is a series that contains x in positive integer powers only; and this series is symbolically the complete general integral by hypothesis.

If two of the independent variables (x, y) take independent minor germs, then the general integral will always admit of being expressed in the form of an infinite series containing both x and y in positive integer powers only.

For in this case

$$\begin{aligned} u &= F(x+g, y+h, z \ldots) \\ &= F(g+x, h+y, z \ldots), \end{aligned}$$

from which the proposition follows by expanding F by Taylor's Theorem.

It is evident the proposition may be extended to all the independent variables that take independent minor germs.

21. If $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right) u = 0$ be understood to represent any linear differential equation of two independent variables (x, y) with constant coefficients, then will each of these variables take an independent minor germ; and consequently the complete general integral of the equation may be fully expressed in any one of the three following equivalent forms,

$$\text{(i)} \ldots u = A + \left(B\frac{x}{1} + B'\frac{y}{1}\right) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \ldots$$

or $$\text{(ii)} \ldots u = P + Q\frac{y}{1} + R\frac{y^2}{1.2} + \ldots$$

in which P, Q, R, ... represent series of the general form

$$A + B\frac{x}{1} + C\frac{x^2}{1.2} + \ldots$$

or $$\text{(iii)} \ldots u = P + Q\frac{x}{1} + R\frac{x^2}{1.2} + \ldots$$

in which P, Q, R, ... represent series of the general form

$$A + B\frac{y}{1} + C\frac{y^2}{1.2} + \ldots$$

When hereafter we assume the first of these forms as representative of the complete general integral of a proposed differential equation, we must remember that the sole authority for the truth of this assumption lies in the fact that we know from the form of the differential equation itself that both x and y take independent minor germs. The second form supposes that y takes a minor germ; and the third form supposes x to take a minor germ.

22. Let $\varpi\left(x, \frac{d}{dx}, \frac{d}{dt}\right)u = 0$, or briefly $\varpi . u = 0$ denote a linear differential equation in which one of the two independent variables (as t) occurs only in the form of a differential symbol of operation.

In this case the general integral is completely represented by the following form,

$$u = F(x, t + g),$$

g being a minor germ of t.

Now if we integrate such a differential equation as the one before us by the method of infinite series, it will sometimes happen that we shall obtain a result which may be represented by the following:

$$u = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots\right),$$

in which A, B, C, ... are independent arbitrary constants; and they are therefore the coefficients of the members of the family of subintegrals of which u is constituted.

Hence we are at liberty to eliminate these arbitrary constants by means of the powers of a germ.

Now $A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots$ is a MacLaurin's infinite series (see Art. 10) and its symbolical equivalent representative will be under a certain restriction of form corresponding to this fact (Art. 11); and the proper form may be found in the following manner.

Equating the two preceding forms of u we have the following symbolical equivalence:

$$F(x, t+g) = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots\right);$$

and the left-hand member being a function of $t + g$, the right-hand member must be so likewise;

$$\therefore\ A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots = f(t+g) \ \ldots\ldots\ldots\ldots\ldots (1).$$

Now $f(t+g) = f(g+t)$, and whichever of these two forms we adopt the result of the expansion of it in a series by Taylor's Theorem must be symbolically equivalent to the left-hand member of equation (1).

Hence

$$A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots = f(g) + f'(g)\cdot\frac{t}{1} + f''(g)\cdot\frac{t^2}{1.2} + \ldots\ (2).$$

But in order that (2) may be a symbolical equivalent of the series on the left-hand, $f(g), f'(g), f''(g), \ldots$ must be different powers of the germ g; a condition that requires the following supposition,

$$f(g+t) = (g+t)^p,$$

where p must be of such a value as shall render the expansion of $(g+t)^p$ an infinite series. Hence p must not be a positive integer.

$$\therefore\ A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots = (g+t)^p, \text{ or} = (t+g)^p.$$

Hence $$u = \psi\left(x, \frac{d}{dt}\right) . (g+t)^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

or $$= \psi\left(x, \frac{d}{dt}\right) . (t+g)^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

There are therefore two forms in which u may be presented, a circumstance which is notable for the following reason.

If p were a positive integer (which it cannot be) these two forms of u would be identical, because in that case the expansions of $(g+t)^p$ and $(t+g)^p$ would be identical, with the exception only that their respective terms would be in a reverse order. But as p is not a positive integer the expansions of $(g+t)^p$ and $(t+g)^p$ are dissimilar, though symbolically equivalent (see Chap. III.).

Hence that the integrals marked (3) and (4) are dissimilar though symbolically equivalent, is due to the circumstance that p is not a positive integer; and further that $(t+g)^p$ and $(g+t)^p$ are symbolically equivalent.

Consequently we have two dissimilar though symbolically equivalent forms in which we may finally present the general integral of the proposed equation, whenever that integral can be found in the form

$$u = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots\right).$$

23. If we expand $(g+t)^p$ in order to eliminate the germ g and obtain the family of subintegrals of which u is constituted, we obtain

$$\text{the first subintegral} = \psi\left(x, \frac{d}{dt}\right) . A = \psi(x, 0).$$

The other subintegrals may all be obtained from this by *integrating* it with respect to t successively (Art. 4 contains an example of this).

But if we expand $(t+g)^p$ the first subintegral will be equal to $\psi\left(x, \frac{d}{dt}\right) . t^p$; and the other members of the family will be obtained from this by successive *differentiations* with respect to t.

As it is always possible to differentiate, and not always possible to integrate, a given function of t, there will be an advantage in using the form

$$\text{first subintegral} = \psi\left(x, \frac{d}{dt}\right) . t^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But here crops up the question,—how are we to assign a proper value to p? for the preceding Article tells us nothing respecting it but that it is not to be a positive integer. It does not even tell us distinctly whether *zero* is to be classed among *positive* or among *negative* integers. There is however no difficulty in seeing that we must assign to p as small a value (apart from its algebraic sign) as possible.

In some degree p is therefore a disposable numerical quantity; and we shall follow the rule of assigning to it the *least* value (apart from algebraic sign) that will enable us *to express the first subintegral* (1) *in finite terms*, for our object is to find the integral of a proposed equation in finite terms.

✓ 24. One possible case must here be noticed. There being nothing to fix a definite value of p in the investigation of Art. 22, in the formula

$$\text{first subintegral} = \psi\left(x, \frac{d}{dt}\right) t^p,$$

if it should ever happen that this leads to a general subintegral of the form $W . \omega^p$, without the necessity of our assigning to p any definite value, then p may be considered to be a germ, and the first subintegral will be *inclusive* of the whole family of subintegrals.

In this particular case therefore we have (by Art. 11),

$$\begin{aligned} u &= W . \omega^p \\ &= W . \phi(\omega). \end{aligned}$$

25. We have said that the smallest possible value (apart from algebraic sign) must be assigned to p in order to obtain

the *first* of the family of subintegrals, and that from the subintegral so found all the other members of the family may be obtained by differentiation with $\frac{d}{dt}$.

From this it is obvious that if we can obtain a finite subintegral by assigning to p a value which is not the least possible (apart from sign) the subintegral so obtained will be one of the family of subintegrals; and we may ascend from it to the *first* subintegral by successive integrations with regard to t. We shall know when we have arrived at the first by the circumstance that we have arrived at a subintegral which is not integrable in finite terms.

26. The general principle that we shall adopt in the integration of linear differential equations is that of taking advantage of any peculiarity that may be perceived to exist in their forms, favoring the introduction of germs into their integrals; for as an integral that is perfectly general cannot be made more general, the introduction of a germ, though it may affect the generality of an integral that is not perfectly general, cannot make it less general; but on the contrary every germ introduced brings it one step nearer to perfect generality.

When therefore a differential equation is proposed for integration we begin by changing (if necessary) the dependent and independent variables (see Chap. IV.) with the object of bringing the equation to its simplest form, or to a form which will enable us to detect the possible existence of germs in the integral.

✓ The preceding Articles will have made it evident that it would be a great point gained if the reduction and transformation can be carried on till we have arrived at a form in which one at least of the independent variables shall occur only in the form of a differential symbol of operation, for such a variable will take a minor germ. The following will illustrate the method of proceeding with such an equation, and will also be useful for reference.

27. To integrate $\frac{du}{dt} = \varpi\left(x, y, \ldots \frac{d}{dx}, \frac{d}{dy}, \ldots\right)u$, or simply $= \varpi \, . \, u$.

This equation allows t to take a minor germ, and therefore (Art. 21) the following will be a perfectly general form of its integral:

$$u = P + Q\frac{t}{1} + R\frac{t^2}{1.2} + S\frac{t^3}{1.2.3} + \ldots$$

in which $P, Q, R, \ldots$ are functions not of t but of the independent variables that occur in the operating function ϖ.

Substitute this form of u in the proposed equation $\frac{du}{dt} = \varpi \, . \, u$;

$$\therefore \; u = \left(1 + \frac{t}{1}\varpi + \frac{t^2}{1.2}\varpi^2 + \frac{t^3}{1.2.3}\varpi^3 + \ldots\right)P.$$

28. To integrate $\frac{d^2u}{dt^2} = \varpi\left(x, y, \ldots \frac{d}{dx}, \frac{d}{dy}, \ldots\right) u$, or briefly, $= \varpi \, . \, u$.

By the same method as the above, and with the additional consideration that this equation allows us to write in any integral jt instead of t, we obtain the following general form of integral:

$$u = \left(1 + \frac{t^2}{1.2}\varpi + \frac{t^4}{4!}\varpi^2 + \ldots\right)P$$
$$+ j\left(\frac{t}{1} + \frac{t^3}{3!}\varpi + \frac{t^5}{5!}\varpi^2 + \ldots\right)Q,$$

in which P, Q are independent functions of the variables contained in ϖ.

The two serial members of u are independent subgeneral integrals; and their independence is due to the circumstance that $\frac{d}{dt}$ occurs in the differential equation only in the form $\left(\frac{d}{dt}\right)^2$, and their independence is secured symbolically by the existence of the ambiguous symbol j in the latter of them.

It will be noticed that one of the subgeneral integrals contains even powers of t only, and the other odd powers only. But we are at liberty to construct out of them, by addition and subtraction, two other equivalent subsequent integrals, each of which shall contain both the odd and the even powers of t.

The following form of differential equation, though it ✓ belongs to the case of two independent variables only, will be found important, for many equations that occur in physical enquiries can be made to depend upon it.

29. To integrate $\phi\left(\frac{d}{dt}\right)u = \varpi\left(x, \frac{d}{dx}\right)u$.

By the usual method of integration by series the integral of this equation can generally be obtained in a form equivalent to the following:

$$u = \psi\left(x, \frac{d}{dt}\right)T,$$

in which T is an arbitrary function of t.

Now since the proposed equation allows t to take a minor germ,

$$\therefore\ T = A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots$$

in which $A, B, C, \dots$ are the arbitrary coefficients of the members of the family of subintegrals which constitute u. Their places may therefore (Art. 13) be supplied by the powers of an extemporized germ m;

$$\therefore\ u = \psi\left(x, \frac{d}{dt}\right)\left(1 + \frac{mt}{1} + \frac{m^2t^2}{1.2} + \dots\right)$$

$$= \psi\left(x, \frac{d}{dt}\right)\epsilon^{mt}$$

$$= \epsilon^{mt}\psi(x, m) = \epsilon^{mt}X.$$

✓ When X is found by the substitution of this value of u in the proposed equation, then u is known from the equation,

$$u = \epsilon^{mt} X.$$

30. Let $\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) u = 0$ be a linear partial differential equation of any number of independent variables and having constant coefficients; then will

$$U = A\epsilon^{Lx+My+Nz+\ldots}$$

be an integral of it; $L, M, N, \ldots$ being germs subject only to the following equation of condition,

$$0 = \varpi(L, M, N, \ldots).$$

For

$$\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) U = \varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) \epsilon^{Lx+My+Nz+\ldots}$$
$$= \epsilon^{Lx+My+Nz+\ldots}\, \varpi(L, M, N, \ldots).$$

Now as $L, M, N, \ldots$ are germs, we are at liberty to assume such a relation to exist among them as will render the right-hand member of this equation equal to zero; and the only condition necessary for that purpose is $\varpi(L, M, N, \ldots) = 0$. Hence subject to this condition U represents a quantity which satisfies the proposed equation.

We have not said that U is the general integral of the equation; but as it contains independent germs it needs must be one of a great degree of generality. As a matter of fact it fails to be the general integral in such cases only as are distinguished by the recurrence of one or more of the operative factorials into which $\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right)$ can in some cases be resolved.

We shall be careful to prove the perfect generality of U in every case in which we shall use it; and then only shall we cite it as *the general exponential* integral.

31. The germs $L, M, N, \ldots$ being of the nature of general germs are liable to contain imaginary quantities; it will some-

times be desirable to express the general exponential integral in terms of *real germs* only. Let us therefore assume their forms to be

$$L+il,\quad M+im,\quad N+in,\ \ldots\ldots$$

in which $L, M, N, \ldots\, l, m, n, \ldots$ are all real quantities.

By this change the exponential integral takes the following form

$$U=A\epsilon^{K+iI}=A\epsilon^{K}\cos I+B\epsilon^{K}\sin I$$

$$=A\epsilon^{K}\cos(I+B);$$

in which $K=Lx+My+Nz+\ldots$

and $I=lx+my+nz+\ldots$

and the germs $L, M, N, \ldots\, l, m, n, \ldots$ are subject to the two equations of condition into which the following equation necessarily divides itself,

$$0=\varpi\,(L+il,\ M+im,\ N+in,\ \ldots).$$

32. If c be a germ contained in an integral U of a linear differential equation $\varpi\,.\,u=0$, containing any number of independent variables and its coefficients not being necessarily constant; then will $\frac{dU}{dc}, \frac{d^2U}{dc^2}, \frac{d^3U}{dc^3}, \ldots$ and generally $\phi\left(\frac{d}{dc}\right)U$ be integrals of $\varpi\,.\,U=0$.

The function ϕ is conditioned by the equation $\phi\left(\frac{d}{dc}\right)0=0$.

Now c being a germ is not contained in ϖ; and therefore c and ϖ are commutative symbols. Also $\varpi\,.\,U=0$.

$$\therefore\ 0=\phi\left(\frac{d}{dc}\right)0=\phi\left(\frac{d}{dc}\right).\,\varpi U=\varpi\,.\,\phi\left(\frac{d}{dc}\right)U.$$

Hence $\phi\left(\frac{d}{dc}\right)U$ is an integral of $\varpi\,.\,u=0$.

33. The following indicates the possible existence of quasi-minor germs in some cases.

Suppose we have before us a linear equation of the following form,

$$0 = \varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, ax + by + cz, \; a'x + b'y + c'z\right),$$

in which a, b, c, a', b', c' are definite constants.

It is evident that we may, without affecting this equation at all, write $x + g, y + h, z + k$ for x, y, z respectively, provided the values of g, h, k are restricted by the two following conditions,

$$0 = ag + bh + ck, \text{ and } 0 = a'g + b'h + c'k.$$

Now as g, h, k are subject to these two linear conditions only, each of them may be described as a definite multiple of some one indefinite quantity l, which we may designate an independent germ. This independent germ will be divided among the three independent variables in certain definite proportions, and be to each of them a minor germ, or rather a quasi-minor germ; for we have defined a minor germ (Art. 8) as belonging exclusively to an individual independent variable.

Major Germs and Homogeneity.

34. By means of major germs we may extend the usual definition of homogeneity in the following manner.

If a mathematical expression $F(x, y, z, \ldots)$ be of such a form that when $m^\alpha x, m^\beta y, m^\gamma z, \ldots$ are written in it for $x, y, z \ldots$ the germ m becomes a mere factor or coefficient of the whole; i.e. if the following form of expression holds good,

$$F(m^\alpha x, m^\beta y, m^\gamma z, \ldots) = m^p F(x, y, z, \ldots),$$

in which $\alpha, \beta, \gamma \ldots$ have definite values; then we say that $F(x, y, z, \ldots)$ is a homogeneous expression of p dimensions.

We may also say that $x, y, z, \ldots$ are respectively of the dimensions $\alpha, \beta, \gamma, \ldots$ and we shall speak of m as being a major germ in this case.

The following proposition will be found very important in future operations.

35. Every homogeneous linear partial differential equation, whether its coefficients be, or be not, constant, will have all its

subintegrals (that are due to the elimination of a major germ) homogeneous according to the above definition; and they will all be of different dimensions.

Let $u = F(x, y, z, \ldots)$ be the integral of a homogeneous differential equation. Then since a general integral is not affected as to its generality by any change of the independent variables which does not affect the differential equation, we may write $m^\alpha x$, $m^\beta y$, $m^\gamma z$, ... in both the equation and its integral without affecting them;

$$\therefore\ u = F(m^\alpha x, m^\beta y, m^\gamma z, \ldots) \ldots\ldots\ldots\ldots\ldots(1),$$

and by differentiation of this we obtain the following equation,

$$\left(\alpha x \frac{d}{dx} + \beta y \frac{d}{dy} + \gamma z \frac{d}{dz} + \ldots\right) u = m \frac{du}{dm} \ldots\ldots\ldots (2).$$

Now expand the right-hand member of equation (1) in powers of m;

$$\therefore\ u = Pm^p + Qm^q + Rm^r + \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

in which P, Q, R ... are functions of x, y, z ... but not of m; they in fact constitute the family of subintegrals due to the elimination of the germ m.

Hence each of them (i.e. of P, Q, R ...) is an integral of the proposed homogeneous equation; and consequently each term of (3) will satisfy the equation (2).

Taking the first term Pm^p and substituting it in (2) we find

$$\alpha x \frac{dP}{dx} + \beta y \frac{dP}{dy} + \gamma z \frac{dP}{dz} + \ldots = pP \ldots\ldots\ldots\ldots(4),$$

the meaning of which equation is, that the subintegral P is homogeneous and of p dimensions.

In the same way we learn that Q, R, ... are homogeneous subintegrals of q, r, ... dimensions respectively.

The members of the family of subintegrals obtained by the elimination of m have therefore this common property,—*they are all homogeneous;* but being of different dimensions their sum, i.e. the general integral which contains them all, is not homogeneous.

Homogeneity is, therefore, the distinctive feature of a subintegral.

36. As a major germ generally (though not always) belongs to at least two independent variables, if a proposed differential equation contains more than two such variables it may admit of more than one independent major germ; or if it admits of one only, there may then be some independent variables that do not take a major germ at all.

Hence it may happen that a differential equation may be homogeneous with regard to only a portion of its independent variables: and being homogeneous it may be of different dimensions in reference to its different major germs.

Thus in the equation $\frac{d^2u}{dxdy} = \frac{du}{dt}$, we may write mx, my, m^2t for x, y, t respectively, and the equation is therefore homogeneous.

Or we may write nx, nt for x and t, and consider the equation homogeneous with respect to x and t. So it is homogeneous with respect to y and t. And we may write lx for x, and $l^{-1}y$ for y. But all these results are included when we write lmx for x, $l^{-1}ny$ for y, and mnt for t, there being three germs involved in this case. This is therefore the most general assumption of major germs; and it implies that the equation is independently homogeneous with regard to x, y, t^2; and to x, t; and y, t. It therefore possesses a triple homogeneity; and to obtain general results all three must be taken account of.

It will now be manifest that the existence of major and minor germs can oftentimes be discovered prior to integration from the form of the proposed differential equation by mere inspection. We shall see hereafter, however, that there may be possible major germs which are not so easily discovered.

And it is always to be remembered that we are at liberty to introduce into a known integral any possible germs, and that the result will be still an integral of the proposed equation, which may be thereby rendered one of increased generality.

CHAPTER III.

ON SYMBOLICAL EQUIVALENCE.

37. We consider the elementary quantities and magnitudes with which we have to do as being measurable by numbers; and an essential property of every such quantity or magnitude is, that "the whole is greater than a part of it."

Zero, which is usually denoted by the symbol 0, we consider to be "the negation of quantity or magnitude." The absence or negation of a quantity cannot be divided into parts; and what has no existence cannot be treated as having properties.

But zero, though non-existent as a measurable quantity, admits of symbolical representation by means of real quantities in an infinite variety of ways; as for example,

$$0 = x - x;\ 0 = 1 + \cos 2x - 2\cos^2 x;$$

$$0 = \left(\frac{d^2}{dx^2} + 1\right)\epsilon^{\iota x};\quad 0 = \left(\frac{d}{dx} - \frac{d}{dy}\right)\phi(x+y).$$

These are called equations, but we here speak of their right-hand members as the symbolical equivalents of zero; and hence the mathematical sign (=) is to be understood not as always denoting numerical equality, since zero is not a number, but as (in such cases as these) denoting symbolical equivalence.

Also such a question as this,—find the integral of $\frac{du}{dx} + \frac{du}{dy} = 0$, may be enunciated in the following equivalent form,—find the most general form of u in terms of x, y which will render the following equation a symbolical equivalence $\left(\frac{d}{dx} + \frac{d}{dy}\right)u = 0$.

The reader will kindly keep in mind, whenever he finds the sign (=) connecting two quantities, or two steps in an investigation, which are not equal algebraically, that that sign is in this case to be read as signifying *symbolical* or *integral equivalence.* We might preserve at every step, and in every equation, *both algebraic and symbolic* equivalence, but this would have to be done oftentimes at an inconvenient expenditure of time and new algbraic symbols. After a little practice no inconvenience will be found in the system employed in these investigations which are chiefly about integrals. The sign (=) has three meanings:—algebraic equality;—symbolical equivalence;—and equal in generality as integrals.

38. For a reason analogous to that which leads us to reject zero as a numerical quantity we reject *infinity,* for it cannot be numerically increased by addition nor diminished by subtraction, since it is not measurable by numbers.

Nevertheless there is a case of infinitude which can be dealt with to advantage, viz., the case of series the number of whose terms is infinite.

Infinite series are of two kinds:—

1. A series may have a first term but no last term; or, in other words, it may have a beginning but no end.

2. A series may have neither a first term nor a last term.

$1+2+2^2+2^3+\ldots$ ad infin. is an example of the former kind, and $\ldots+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2}+1+2+2^2+2^3+\ldots$ of the latter kind.

39. The meaning of the word equivalence which it will be necessary to attach to the sign (=) in some of the subsequent articles is so unusual that we shall add a few more illustrations.

The late Professor De Morgan proposed the following equation for solution,

$$x=2x.$$

If x be in this equation a numeral quantity, divide both sides of the equation by x. Then on the ground that if equals in the following manner, with the object of presenting the integral in the briefest possible form.

$$\begin{aligned} u &= A\left(1+\frac{x^2}{1.2}+\ldots.\right)+B\left(\frac{x}{1}+\frac{x^3}{1.2.3}+\ldots.\right) \\ &= A\,(\epsilon^x+\epsilon^{-x})+B\,(\epsilon^x-\epsilon^{-x}) \\ &= A\epsilon^x+B\epsilon^{-x} \\ &= A\epsilon^{jx}. \end{aligned}$$

This is our final result, and simple as it is, it is perfectly equivalent as an integral to the two infinite series which constitute the entire value of u. In deducing it from those series the arbitrary constants have suffered changes of identity at every step, but we have been careful to preserve their only essential quality, that they are arbitrary constants all through the process of reduction.

Had the equation to be integrated been $\frac{d^2u}{dx^2}+u=0$, our result would have been

$$u=A\epsilon^{\iota x}.$$

It will be seen from the above example that we shall hereafter feel at liberty to use the sign $(=)$, as denoting integral equivalence; and that in so using it we shall consider not the identity of the quantities denoted by A, B, C, ..., but merely take care that each shall preserve its only essential quality, viz., that it denotes a perfectly arbitrary and independent quantity.

43. If $f(x)$ be expanded in an infinite series it is usual to represent the result thus,

$$f(x)=Ax^\alpha+Bx^\beta+Cx^\gamma+\ldots \text{ ad inf.}$$

Professor De Morgan proposed that the left-hand member should be denominated the *Involopment* of the right-hand member. We shall adopt this designation. It has been usual to speak of $f(x)$ as the *sum* of the series, but unless the series be convergent this designation is incorrect.

When we meet with two symbolically equivalent series, if we can find the involopment of one of them we shall use that series in preference to the other without reference to its convergence or non-convergence.

44. A series that has no last term may possess properties not possessed by the sum of any number of its terms. Take the following example:

$$U = 1 - x + x^2 - x^3 + \dots \text{ ad inf. } \dots\dots\dots\dots (1).$$

If we multiply this series by x and subtract the product from unity it remains unchanged; and this is not a property of the series continued to n terms only. Hence the following is true of the infinite series only, viz.

$$U = 1 - xU;$$

$$\therefore\ U = \frac{1}{1+x}.$$

This is the invelopment of the series; i.e. it represents the whole infinite series, and if it be expanded according to the usual rules it will be found to produce the whole series.

But $\frac{1}{1+x} = \frac{1}{x+1}$, and the latter expression is the invelopment of the following infinite series,

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \dots \text{ ad inf. } \dots\dots\dots\dots (2).$$

Now since the invelopments of these respective series are symbolically and algebraically equal, we say that the following is both a symbolical and an integral equivalence,

$$1 - x + x^2 - \dots \text{ ad inf.} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \dots \text{ ad inf.};$$

and therefore in reducing an integral to its simplest or most manageable form we should not hesitate, if necessary to secure ultimate success, to introduce this equivalence, or any other which rests on the same basis.

45. We shall now generalize the above results by shewing that the two following infinite series are symbolically equivalent,

$$A + Bx + Cx^2 + \dots = A + Bx^{-1} + Cx^{-2} + \dots$$

in which A, B, C, ... are constant quantities, definite or indefinite.

Instead of C, D, E, ... in the left-hand member write respectively $C' - 2B$, $D' - 4C' + 3B$, $E' - 6D' + 10C' - 4B$, &c.

$$\begin{aligned} \therefore\ A + Bx + Cx^2 + Dx^3 + \dots = A &+ B(x - 2x^2 + 3x^3 - 4x^4 + \dots) \\ &+ C'(x^2 - 4x^3 + 10x^4 - \dots) \\ &+ D'(x^3 - 6x^4 + \dots) \\ &+ \&c. \end{aligned}$$

$$= A + \frac{Bx}{(1+x)^2} + \frac{C'x^2}{(1+x)^4} + \frac{D'x^3}{(1+x)^6} + \dots$$

$$= A + \frac{Bx^{-1}}{(1+x^{-1})^2} + \frac{C'x^{-2}}{(1+x^{-1})^4} + \frac{D'x^{-3}}{(1+x^{-1})^6} + \dots$$

$$= A + Bx^{-1} + Cx^{-2} + Dx^{-3} + \dots$$

The validity of this investigation depends entirely on the series being infinite, and it cannot hold good for n terms, with the single exception of $n = 1$.

It is to be noticed also that the quantities C', D', E', ... are used in the proof merely as artificial means of distributing the terms of the left-hand series into groups suitable for our purpose. And it is obvious that a different grouping would have led us to another type of symbolical equivalence; as will be seen in the following Article.

46. To shew that the two following infinite series are symbolically equivalent to each other;

$$A + Bx + Cx^2 + \dots = x^p(A + Bx^{-1} + Cx^{-2} + \dots),$$

the index p being subject to the sole condition that it must not be a positive integer.

Instead of B, C, D, ... in the left-hand series substitute the following quantities,

$$B = B' + \frac{p}{1}A,$$

$$C = C' + \frac{p-2}{1}B' + \frac{p(p-1)}{1.2}A,$$

$$D = D' + \frac{p-4}{1} C' + \frac{(p-2)(p-3)}{1 \,.\, 2} B' + \frac{p(p-1)(p-2)}{1 \,.\, 2 \,.\, 3} A,$$

&c. = &c.,

the law of these substitutions being obvious, and requiring, as the series are infinite, that p shall not be a positive integer.

On making these substitutions and proceeding step by step as in the preceding Article we arrive at the following symbolical equivalence,

$$\begin{aligned} A + Bx + Cx^2 + \ldots &= x^p (A + Bx^{-1} + Cx^{-2} + \ldots) \\ &= Ax^p + Bx^{p-1} + Cx^{p-2} + \ldots \end{aligned}$$

47. This result may be presented in the following form,

$$\begin{aligned} A + Bx + Cx^2 + \ldots &= \left\{A + \frac{B}{p} \frac{d}{dx} + \frac{C}{p(p-1)} \left(\frac{d}{dx}\right)^2 + \ldots\right\} x^p \\ &= F\left(\frac{d}{dx}\right) . x^p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

And if A, B, C, ... are absolutely arbitrary and independent, then is also the function $F\left(\frac{d}{dx}\right)$ an arbitrary function of $\frac{d}{dx}$, subject only to $F\left(\frac{d}{dx}\right) 0 = 0$.

The value of this result in the discovery of subintegrals will be seen when we come to the actual integration of equations.

The reader may compare these results with Art. 23.

CHAPTER IV.

THE TRANSFORMATION OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

I. *Two independent variables; coefficients constant.*

48. OUR object in this chapter is to reduce equations to their most simple forms with the view of discovering those forms which present peculiar integrational difficulties.

We may classify any linear differential equation of two independent variables and having constant coefficients under some one of the four following heads,

$$(\alpha)\ldots 0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\beta)\ldots 0 = \frac{d^2u}{dx^2} + (a+b)\frac{d^2u}{dx\,dy} + ab\frac{d^2u}{dy^2} + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\gamma)\ldots 0 = \left(\frac{d}{dx} + \frac{ad}{dy}\right)^2 u + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\delta)\ldots 0 = \left(\frac{d}{dx} + \frac{ad}{dy}\right)^2 u + A\left(\frac{d}{dx} + \frac{ad}{dy}\right) u + Cu.$$

49. To reduce the form (α) assume a new dependent variable v such that

$$u = ve^{-Ay-Bx}.$$

This gives the following reduced form when substituted for u,

$$0 = \frac{d^2v}{dx\,dy} + (C - AB)\,v,$$

which comprehends the two following elementary forms,

$$(1)\ldots\ldots 0 = \frac{d^2\omega}{dx\,dy}, \text{ and } \frac{d^2\omega}{dx\,dy} = \omega \ldots\ldots (2).$$

The former of these presents no integrational difficulty; and the latter we shall integrate in a future Article.

50. To reduce the form (β) we change the dependent variables by assuming two new variables ξ, η such that

$$x = \xi + \eta \text{ and } y = a\xi + b\eta.$$

$$\therefore \frac{du}{d\xi} = \left(\frac{d}{dx} + a\frac{d}{dy}\right)u, \text{ and } \frac{du}{d\eta} = \left(\frac{d}{dx} + b\frac{d}{dy}\right)u,$$

and the reduced equation is

$$0 = \frac{d^2u}{d\xi\,d\eta} + \frac{B - bA}{a - b}\frac{du}{d\xi} + \frac{B - aA}{b - a}\frac{du}{d\eta} + Cu,$$

which being of the form (α) can be reduced to the forms (1) and (2), and therefore furnishes no new integrational difficulty.

51. To reduce form (γ) we assume $x = \xi + \eta$ and $y = a\xi + \frac{B}{A}\eta$;

$$\therefore \frac{du}{d\xi} = \left(\frac{d}{dx} + a\frac{d}{dy}\right)u, \text{ and } \frac{du}{d\eta} = \left(\frac{d}{dx} + \frac{B}{A}\frac{d}{dy}\right)u,$$

and the following is the form of the reduced equation,

$$0 = \frac{d^2u}{d\xi^2} + A\frac{du}{d\eta} + Cu;$$

and by changing the dependent variable, if necessary, by writing $v\epsilon^{m\eta}$ for u, m being such as to satisfy the equation $Am + C = 0$, we obtain the following form,

$$0 = \frac{d^2v}{d\xi^2} + A\frac{dv}{d\eta}.$$

The following are therefore the ultimate forms furnished by form (γ),

$$(3)\ldots\ldots 0 = \frac{d^2\omega}{dx^2} + \omega, \quad \frac{d^2\omega}{dx^2} = \frac{d\omega}{dy} \ldots\ldots (4),$$

and

$$\frac{d^2\omega}{dx^2} = 0 \ldots\ldots (5),$$

of which (4) only presents any new integrational difficulty.

52. To reduce form (δ) we assume $x = \xi + \eta$ and $y = a\xi + b\eta$;

$$\therefore \frac{du}{d\xi} = \left(\frac{d}{dx} + a\frac{d}{dy}\right)u,$$

and the reduced equation is

$$0 = \frac{d^2u}{d\xi^2} + A\frac{du}{d\xi} + Cu,$$

a form which does not contain η $\left(\text{which is equal to } \frac{ax-y}{a-b}\right)$ and presents no integrational difficulty. It is in fact an equation of one independent variable; and consequently, when it is integrated, arbitrary functions of η, or rather of $(ax - y)$ must be used instead of arbitrary constants.

53. Hence gathering together the forms that are of difficult integration we find only the two following,

$$\frac{d^2u}{dx\,dy} = u, \text{ and } \frac{d^2u}{dx^2} = \frac{du}{dy}.$$

These forms it will be our business to integrate in the following chapter.

II. *The case of three independent variables.*

54. We may arrange any equation of this class under some one of the four following heads,

$$(\alpha)\ldots 0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\beta)\ldots 0 = \frac{d^2u}{dx\,dy} + a\frac{d^2u}{dx\,dz} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\gamma)\ldots 0 = \frac{d^2u}{dx\,dy} + a\frac{d^2u}{dx\,dz} + b\frac{d^2u}{dy\,dz} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\delta)\ldots 0 = \frac{d^2u}{dx^2} + \alpha\frac{d^2u}{dy^2} + \beta\frac{d^2u}{dz^2} + a\frac{d^2u}{dx\,dy} + b\frac{d^2u}{dx\,dz} + c\frac{d^2u}{dy\,dz}$$
$$+ A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

55. To reduce the form (α) we assume $u = v\epsilon^{-Ay-Bx-mz}$, where m is a constant that satisfies the equation $mC + AB = K$, and the following is the reduced form of the equation,

$$0 = \frac{d^2v}{dx\,dy} + C\frac{dv}{dz},$$

which includes the two following elementary forms,

$$(1)\ldots\ldots\frac{d^2\omega}{dx\,dy} = 0, \text{ and } \frac{d^2\omega}{dx\,dy} = \frac{d\omega}{dz}\ldots\ldots(2).$$

The former of these presents no integrational difficulty.

56. To reduce the form (β) let x, y, ζ be a new set of independent variables, in which $\zeta = z - ay$. The reduced form of the equation is

$$0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + (C - aB)\frac{du}{d\zeta} + Ku,$$

which coinciding with form (α) introduces no additional integrational difficulty.

57. To reduce form (γ), for u write $ve^{mx+ny+pz}$, the constants m, n, p being such as will satisfy the three following equations:

$0 = A + n + ap,\quad 0 = B + m + bp,\quad$ and $\quad 0 = C + am + bn.$

The reduced equation is the following,

$$0 = \frac{d^2v}{dx\,dy} + a\frac{d^2v}{dx\,dz} + b\frac{d^2v}{dy\,dz} + K'v,$$

in which $K' = Cp - mn + K$.

Let now the independent variables be changed to x, y, ζ where $\zeta = z - ay - bx$. By this means the reduced equation becomes

$$\frac{d^2v}{dx\,dy} = ab\frac{d^2v}{d\zeta^2} + K'v,$$

which includes the two following new elementary forms,

$$(3)\ldots\ldots\frac{d^2\omega}{dx\,dy} = \frac{d^2\omega}{d\zeta^2}, \text{ and } \frac{d^2\omega}{dx\,dy} = \frac{d^2\omega}{d\zeta^2} + \omega\ldots\ldots(4).$$

58. To reduce the form (δ), assume a new set of independent variables ξ, η, ζ such that

$$\xi = x - gy, \quad \eta = y - hz, \quad \text{and} \quad \zeta = z - kx,$$

the constants g, h, k being such as will satisfy the following conditional equations,

$$0 = ag^2 - ag + 1, \quad 0 = \beta h^2 - ch + 1, \quad \text{and} \quad 0 = k^2 - bk + \beta.$$

By these means the form (δ) will be reduced to form (γ), and consequently introduces no new elementary forms.

59. Gathering together the elementary forms which present integrational difficulties, we find that they are the three following:

$$\frac{d^2u}{dx\,dy} = \frac{du}{dz}, \quad \frac{d^2u}{dx\,dy} = \frac{d^2u}{dz^2}, \quad \text{and} \quad \frac{d^2u}{dx\,dy} = \frac{d^2u}{dz^2} + u.$$

In this chapter we are therefore presented with five difficult linear differential equations of the second order with constant coefficients; viz. two when there are two independent variables, and three when there are three independent variables.

CHAPTER V.

INTEGRATION OF EQUATIONS OF TWO INDEPENDENT VARIABLES.

In the preceding chapter we have seen that the two following equations present the only difficulties that are experienced in the integration of linear equations of the second order with constant coefficients. In this chapter we shall bring in the properties of germs to our aid in the task of effecting their complete integration.

60. To integrate $\dfrac{d^2u}{dx^2} = \dfrac{du}{dt}$.

According to Art. 21 the following is a series which may be assumed for the complete integral of this equation,

$$u = P + Q\frac{t}{1} + R\frac{t^2}{1 \,.\, 2} + \dots,$$

which being substituted in the proposed equation gives the following complete form of u,

$$u = \left(1 + \frac{t}{1}\,\frac{d^2}{dx^2} + \frac{t^2}{1 \,.\, 2}\cdot\frac{d^4}{dx^4} + \dots\right)P,$$

where $$P = A + B\frac{x}{1} + C\frac{x^2}{1 \,.\, 2} + \dots,$$

the constants A, B, C, ... which are absolutely arbitrary, being the coefficients of the subintegral constituents of u. Hence we may write $M, M^2, M^3, \ldots$ for them (Art. 13), M being an extemporized germ; and then we shall have the following equivalence,

$$P = A + B\frac{x}{1} + C\frac{x^2}{1 . 2} + \ldots = A\epsilon^{Mx};$$

$$\therefore\ u = A\left(1 + \frac{t}{1}\frac{d^2}{dx^2} + \frac{t^2}{1 . 2}\cdot\frac{d^4}{dx^4} + \ldots\right)\epsilon^{Mx}$$

$$= A\epsilon^{M^2t+Mx}.$$

Thus it is proved, for the proposed equation, that the general exponential integral of Art. 30 is the *complete* integral.

In this form of the general integral the minor germs of x and t are implicitly contained in A, the general coefficient; and M is a general germ, i.e. it is liable to contain both real and imaginary quantities. The major germ is explicitly contained in the integral, on which account the integral takes a form which we may refer to as the major-germ form.

61. In Art. 31 we have shewn the general method of expressing an exponential integral, that contains general germs, in an equivalent integral containing real germs only.

In the integral just found we have merely to write $M + im$ for M; and the following is the form in real germs M, m;

$$\therefore\ u = A\epsilon^{(M^2-m^2)t+Mx}\cos m\,(2Mt + x + B).$$

The minor germs of x and t are in this integral implicitly contained in the arbitrary constants A, B. The form is a major-germ form.

62. To find the integral of $\frac{d^2u}{dx^2} = \frac{du}{dt}$ in a minor-germ form, i.e. in a form which renders the major germ latent in the arbitrary constants of the integral.

From Art. 34 we learn that the subintegrals obtained by the elimination of a major germ will all be homogeneous and of different dimensions. This therefore suggests the following method of obtaining the subintegrals required.

Let V be a function of x and t which is of zero dimensions. The general representative of such a function in the case of the proposed example will be $V = \phi\left(\frac{x}{\sqrt{t}}\right)$, for if Mx, M^2t be written in this for x, t, the germ M will disappear. Denote $\frac{x}{\sqrt{t}}$ by v, and then the following general form of homogeneity will represent any one of the subintegrals,

$$P = t^p V,$$

the dimensions of this subintegral being p.

This being an integral of the proposed equation must satisfy it, and being substituted therein, the following is the resulting equation for the determination of V,

$$\frac{d^2V}{dv^2} + \frac{v}{2}\frac{dV}{dv} = pV.$$

Now we wish to obtain subintegrals in a finite form, or if that be not possible, then in a form that shall give a finite expression for u.

We make use of p (which is disposable) for this purpose; and enquire what value of p will give a finite expression for V. We can see at once that $p = -\frac{1}{2}$ will answer our purpose. The above equation being integrated on this supposition, we find

$$V = A\epsilon^{-\frac{v^2}{4}} + B\epsilon^{-\frac{v^2}{2}}\int \epsilon^{\frac{v^2}{4}}\,dv;$$

$$\therefore P = t^p V = At^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}} + Bt^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}}\int \epsilon^{\frac{v^2}{4}}\,dv.$$

The last term we reject because it is not in a finite form;

$$\therefore\ u = F\left(\frac{d}{dx}, \frac{d}{dt}\right).\,t^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}}.$$

Now the proposed equation shews that $\frac{d}{dt}$ when applied to any integral of the proposed equation is equivalent to $\left(\frac{d}{dx}\right)^2$ applied to the same integral;

$$\therefore\ F\left(\frac{d}{dx},\ \frac{d}{dt}\right)=F\left(\frac{d}{dx}\right);$$

$$\therefore\ u=F\left(\frac{d}{dx}\right).t^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}},$$

and this is the general integral of the proposed equation, in a form that renders the major germ latent. (Here the sign = denotes integral equivalence; and F stands for the *words* "arbitrary function of.")

Lest the reader should have any doubt of the generality of this result we will obtain it in another manner.

63. The proposed equation $\frac{d^2u}{dx^2}=\frac{du}{dt}$ has constant coefficients, consequently the following is by Art. 21 the general assumption for its perfect integral,

$$u=P+Q\frac{x}{1}+R\frac{x^2}{1.2}+S\frac{x^3}{1.2.3}+\ldots,$$

in which P, Q, R, ... are serial functions of t of the general form $A+B\frac{t}{1}+C\frac{t^2}{1.2}+\ldots$

The substitution of this value of u in the proposed equation furnishes the following form of the general integral,

$$\left.\begin{aligned} u=&\left(1+\frac{x^2}{2\,!}\,\frac{d}{dt}+\frac{x^4}{4\,!}\,\frac{d^2}{dt^2}+\ldots\right)P \\ &+\left(\frac{x}{1}+\frac{x^3}{3\,!}\,\frac{d}{dt}+\frac{x^5}{5\,!}\,\frac{d^2}{dt^2}+\ldots\right)Q \end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

These are the two subgeneral integrals; and the former contains only even powers of x, and the latter only its odd powers; and this is due to the fact that $\frac{d}{dx}$ occurs in the proposed equation in the form $\left(\frac{d}{dx}\right)^2$ only.

The first subgeneral

$$= \left(1 + \frac{x^2}{2!}\frac{d}{dt} + \dots\right) P$$

$$= \psi\left(\frac{x^2 d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots\right)$$

$$= \psi\left(\frac{x^2 d}{dt}\right)(t+g)^p \text{......see Art. 22,}$$

$$= \left\{1 + \frac{p}{2!}\left(\frac{x^2}{t+g}\right) + \frac{p(p-1)}{4!}\left(\frac{x^2}{t+g}\right)^2 + \dots\right\}(t+g)^p.$$

Our wish is to obtain the subgeneral integrals in a finite form, and therefore we now ask what value of p will enable us to find the involopment of this infinite series. There is no particular difficulty in seeing that $p = -\frac{1}{2}$ will enable us to do this;

$$\therefore \text{ first subgeneral} = \epsilon^{-\frac{x^2}{4(t+g)}}(t+g)^{-\frac{1}{2}}$$

$$= F\left(\frac{d}{dt}\right).\, t^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}} \text{......(Art. 19).}$$

And from this we can deduce the *form* of the second subgeneral integral.

$$\text{For} \quad F\left(\frac{d}{dt}\right).\, t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}} = \left(1 + \frac{x^2}{2!}\frac{d}{dt} + \frac{x^4}{4!}\frac{d^2}{dt^2} + \dots\right)P.$$

Differentiate with $\frac{d}{dx}$.

$$\therefore F\left(\frac{d}{dt}\right)\cdot\frac{d}{dx}t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}} = \left(\frac{x}{1} + \frac{x^3}{3!}\frac{d}{dt} + \frac{x^5}{5!}\frac{d^2}{dt^2} + \dots\right)\cdot\frac{d}{dt}P$$

$$= \left(\frac{x}{1} + \frac{x^3}{3!}\frac{d}{dt} + \dots\right)\left(B + C\frac{t}{1} + D\frac{t^2}{1.2} + \dots\right).$$

Now the right-hand member of this is of precisely the same *form* and *generality* as the second subgeneral integral in (1);

$$\therefore \text{ second subgeneral} = f\left(\frac{d}{dt}\right).\frac{d}{dx}t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}}$$

$$= f\left(\frac{d}{dx}\right)t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}},$$

with the understanding that this integral shall contain only odd powers of x.

Hence if we gather the two subgeneral integrals together we have two terms, of which one contains only even powers of x, and the other only odd powers;

$$\therefore\ u = \left\{F\left(\frac{d}{dt}\right) + f\left(\frac{d}{dx}\right)\right\} . \, t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}}$$

$$= F\left(\frac{d}{dx}\right) . \, t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}},$$

which agrees with the result obtained in the previous Article.

64. Hence we have found the two following forms of the general integral of the equation $\frac{d^2u}{dx^2} = \frac{du}{dt}$,

$$(1) \ldots u = A\epsilon^{M^2t+Mx},$$

in which the major germ M is explicit, and the minor germs latent; and

$$(2) \ldots u = F\left(\frac{d}{dx}\right) . \, t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}},$$

in which both the major and minor germs are latent in the general operative function $F\left(\frac{d}{dx}\right)$;

$$\therefore\ A\epsilon^{M^2t+Mx} = F\left(\frac{d}{dx}\right) . \, t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}} .$$

It will not be forgotten by the reader, that when (=) does not denote algebraic equality, it denotes the words "symbolical equivalence."

65. To integrate $\frac{d^2u}{dx\,dy} = u$.

Both x and y take minor germs. Hence the general integral can be completely expressed in a series containing only positive integer powers of x and y.

The following may therefore be assumed as a general form of u,

$$u = P + Q\frac{y}{1} + R\frac{y^2}{1.2} + S\frac{y^3}{1.2.3} + \dots,$$

P, Q, R, ... being serial functions of x of the general form

$$A + B\frac{x}{1} + C\frac{x^2}{1.2} + \dots$$

Substitute this form of u in the proposed equation; and the following is the result (in which we take the liberty of using $\frac{dx}{d}$ for the symbol of integration),

$$\begin{aligned} u &= \left\{1 + \frac{y}{1}\frac{dx}{d} + \frac{y^2}{1.2}\left(\frac{dx}{d}\right)^2 + \dots\right\} P \\ &= \psi\left(y\frac{dx}{d}\right)\left(A + B\frac{x}{1} + C\frac{x^2}{1.2} + \dots\right) \\ &= \psi\left(y\frac{dx}{d}\right) . \epsilon^{cx} \text{ ...... Art. (29)} \\ &= A\left(1 + \frac{y}{1}.\frac{1}{c} + \frac{y^2}{1.2}.\frac{1}{c^2} + \dots\right)\epsilon^{cx} \end{aligned}$$

(1) ...... $= A\epsilon^{cx+c^{-1}y}$, c being a general germ.

Hence the exponential integral

$$u = A\epsilon^{Mx+Ny}, \text{ subject to } MN = 1,$$

is perfectly general in the example before us in this Article.

We may obtain the first subintegral in the following manner by the elimination of the germ c from (1).

The form of the proposed equation shews that the product (xy) is of zero dimensions. Let $v^2 = xy$, and let V be a function of v. We may assume the following as the general representative of subintegrals,

$$P = x^p V.$$

This being substituted in the proposed equation gives the following for the determination of V,

$$\frac{d^2V}{dv^2} + \frac{2p+1}{v}\frac{dV}{dv} - 4V = 0.$$

This will be integrable in a finite form if we assume $2p+1=0$, and therefore $p=-\frac{1}{2}$;

$$\therefore\ V = A\epsilon^{2v} + B\epsilon^{-2v}$$
$$= A\epsilon^{2jv},$$

and the first subintegral $P = x^{-\frac{1}{2}}V$

$$= x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}};$$

$$\therefore\ u = F\left(\frac{d}{dx}, \frac{d}{dy}\right) . x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} \ \ldots\ldots\ldots\ldots\ldots\ (2).$$

Now it appears from the form of the proposed equation that the symbolical product of $\frac{d}{dx}$ and $\frac{d}{dy}$ is equivalent to unity, when applied to any integral of that equation; the above form of u may therefore be presented in the following equivalent form,

$$u = F\left(\frac{d}{dx}\right) . x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} + f\left(\frac{d}{dy}\right) . y^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} \ldots\ldots\ldots (3).$$

The following equation is obviously true, and it gives rise to the latter of these two subgeneral integrals,

$$\frac{d}{dy} . x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} = y^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}}.$$

It has therefore been proved above that the elimination of the major germ c from the exponential integral

$$u = A\epsilon^{cx+c^{-1}x} \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

gives the following form of the first subintegral, to which we shall often have occasion to refer,

$$P = x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}}.$$

Now the integral (4) is expressed in terms of c as a general germ; but we may express it in real germs by writing

$c(\cos m + i \sin m)$ for c,

and $c^{-1}(\cos m - i \sin m)$ for c^{-1},

in which new forms of the germs, c and m are to be considered real germs.

Let $K = cx + c^{-1}y$, and $I = cx - c^{-1}y$, then the integral (4) will take the following equivalent general form of expression in real germs,

$$u = A\epsilon^{K\cos m}\cos(I\sin m + B) \,\ldots\ldots\ldots\ldots\ldots (5).$$

66. The various integrals of $\dfrac{d^2u}{dx\,dy} + u = 0$ may be deduced from the two preceding Articles by writing therein $-y$ for y.

$$\therefore\ u = A\epsilon^{Mx+Ny} = F\left(\frac{d}{dx}, \frac{d}{dy}\right).x^{-\frac{1}{2}}\cos(2\sqrt{xy} + B)$$

subject to $MN + 1 = 0$.

The following Article is introduced for the purpose of future reference.

67. To change the independent variables of the expression

$$\frac{d^2u}{dx\,dy}.$$

Let ξ and η the new independent variables be such that

$$\frac{d}{dx} = \frac{d}{d\xi} + a\frac{d}{d\eta}, \text{ and } \frac{d}{dy} = \frac{d}{d\xi} + b\frac{d}{d\eta}.$$

These assumptions require that a, b shall not be equal.

$$\therefore\ \xi = x + y, \text{ and } \eta = ax + by,$$

and also, $$x = \frac{\eta - b\xi}{a - b}, \text{ and } y = -\frac{\eta - a\xi}{a - b}.$$

$$\therefore\ \frac{d^2u}{dx\,dy} = \left(\frac{d}{d\xi} + a\frac{d}{dy}\right)\left(\frac{d}{d\xi} + b\frac{d}{d\eta}\right)u$$

$$= \frac{d^2u}{d\xi^2} + (a + b)\frac{d^2u}{d\xi\,d\eta} + ab\frac{d^2u}{d\eta^2} \,\ldots\ldots\ldots\ldots (1).$$

Also $$(a - b)^2 xy = -(\eta - a\xi)(\eta - b\xi)$$

$$= -(\eta^2 - \overline{a + b}\,\eta\xi + ab\xi^2) \,\ldots\ldots\ldots\ldots (2).$$

68. To integrate

$$\frac{d^2u}{d\xi^2} + (a + b)\frac{d^2u}{d\xi\,d\eta} + ab\frac{d^2u}{d\eta^2} = u.$$

We deduce the required integral from Art. 65 by writing therein the above values of x and y in terms of ξ and η.

$$\therefore\ u = F\left(\frac{d}{d\xi} + a\frac{d}{d\eta}\right).(\eta - b\xi)^{-\frac{1}{2}}\epsilon^{\frac{2i}{a-b}(\eta^2 - a + b\eta\xi + ab\xi^2)^{\frac{1}{2}}}$$

$$+ f\left(\frac{d}{d\xi} + b\frac{d}{d\eta}\right).(\eta - a\xi)^{-\frac{1}{2}}\epsilon^{\frac{2i}{a-b}(\eta^2 - \overline{a+b}\eta\xi + ab\xi^2)^{\frac{1}{2}}}.$$

69. To integrate the equation $\left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n u = 0$, in which the compound operative symbol $\left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)$ is repeated n times.

Changing the dependent variable, assume either $u = \epsilon^{\jmath mx} Y$, or $u = \epsilon^{m^2y} X$.

We begin with the former (m being a general germ).

$$\therefore\ 0 = \left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n \epsilon^{\jmath mx}\, Y$$

$$= \epsilon^{\jmath mx}\left(m^2 - \frac{d}{dy}\right)^n Y,$$

$$\therefore\ 0 = \epsilon^{m^2y}\left(\frac{d}{dy}\right)^n . Y\epsilon^{-m^2y},$$

$$\therefore\ 0 = \left(\frac{d}{dy}\right)^n . Y\epsilon^{-m^2y},$$

$$\therefore\ Y\epsilon^{-m^2y} = A + B\frac{y}{1} + C\frac{y^2}{1.2} + \ldots + N\frac{y^{n-1}}{(n-1)!}$$

$$= A(h+y)^{n-1},$$

h being a minor germ of y.

$$\therefore\ u = \epsilon^{\jmath mx}\, Y$$

$$= A\epsilon^{\jmath mx + m^2y}(h+y)^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Had we taken the form $u = \epsilon^{m^2y} X$ we should have found

$$0 = \left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n \epsilon^{m^2y} X,$$

$$\therefore\ 0 = \left(\frac{d^2}{dx^2} - m^2\right)^n X$$

$$= \left(\frac{d}{dx} - m\right)^n \left(\frac{d}{dx} + m\right)^n X,$$

which is equivalent to the following integrals,

$$\left(\frac{d}{dx}-m\right)^n X=0, \text{ and } \left(\frac{d}{dx}+m\right)^n X=0,$$

$$\therefore \left(\frac{d}{dx}\right)^n . X\epsilon^{-mx}=0, \text{ and } \left(\frac{d}{dx}\right)^n . X\epsilon^{mx}=0,$$

$$\therefore u=\epsilon^{m^2y}\{A\epsilon^{mx}(x+g)^{n-1}+B\epsilon^{-mx}(x+l)^{n-1}\} \text{ .... (2)},$$

which agrees with (1); since $A\epsilon^{jmx}$ represents both $A\epsilon^{mx}$ and $B\epsilon^{-mx}$.

A similar method of treatment will succeed with the equation

$$\left(\frac{d^2}{dx\,dy}-1\right)^n u=0.$$

We have hitherto confined ourselves to equations with constant coefficients; but in the following examples the coefficients are functions of one of the independent variables.

70. To integrate

$$\frac{d^2u}{dx\,dy}=\frac{a}{x}\frac{du}{dx}+\frac{b}{x}\frac{du}{dy}.$$

In this equation x and y can take a major germ; and y can take a minor germ also.

Hence changing the dependent variable we assume the following general form for the integral of this equation,

$$u=\epsilon^{my}X,$$

m being the major germ, and X being a function of x.

This value of u being substituted in the proposed equation gives the following for the determination of X:

$$\left(x-\frac{a}{m}\right)\frac{dX}{dx}=bX,$$

$$\therefore X=A\left(x-\frac{a}{m}\right)^b$$

$$=A(mx-a)^b,$$

$$\therefore u=A(mx-a)^b\epsilon^{my} \text{ .... (1)}.$$

In this integral the minor germ is latent, and the major germ is explicitly involved in it. Also m is a general germ.

Again, to find the general integral in a form which renders the major germ latent.

We assume $x^p V$ as the general type of the subintegrals; V being a function of v, and $v = \frac{y}{x}$.

Substituting $x^p V$ for u in the proposed equation we find the following equation for the determination of V in a finite form;

$$\frac{d}{dv}\left(v\frac{dV}{dv}\right) + (b - p - av)\frac{dV}{dv} + apV = 0.$$

That this may be integrable immediately the following condition must be satisfied; $ap = -a$.

Consequently in the general case $p = -1$; but when $a = 0$, p will be subject to no condition. $a = 0$ is therefore an exceptional case.

$$\therefore \frac{d}{dv}\left(v\frac{dV}{dv}\right) + (1 + b - av)\frac{dV}{dv} - aV = 0.$$

This equation being integrated, gives the following as the first subintegral,

$$P = \frac{V}{x} = \frac{Ax^b}{y^{b+1}}\epsilon^{\frac{ay}{x}} + \frac{Bx^b}{y^{b+1}}\epsilon^{\frac{ay}{x}}\int v^b\epsilon^{-av}dv \ldots\ldots\ldots\ldots (2),$$

and $$u = F\left(\frac{d}{dy}\right)(x^{-1}V).$$

Thus the proposed equation is completely integrated in a form that renders the major germ m latent; but the term multiplied by B will not be in a finite form, and will therefore have to be rejected, unless b be a positive integer.

We will now take the exceptional case, viz. when $a = 0$.

71. To integrate $\frac{d^2u}{dx\,dy} = \frac{b}{x}\frac{du}{dy}$.

$$\therefore u = x^b F(y) + f(y).$$

CHAPTER VI.

EQUATIONS NEARLY RELATED TO LAPLACE'S EQUATION.

Coefficients constant.

72. To integrate $\frac{d^2u}{dx^2}=\frac{d^2u}{dt^2}$.

Let M be a general germ.

$$\therefore\ u=A\epsilon^{M(x+jt)}$$

$$=\phi(x+jt)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Expressed in terms of real germs only, we have by the method of Art. 31,

$$u=A\epsilon^{M(x+jt)}\cos m(x+jt+B)\ldots\ldots\ldots\ldots\ldots(2),$$

M and m being in this form independent real germs.

To obtain the integral from which major germs are eliminated, let us assume $v=\frac{x}{t}$, and $V=\phi(v)$. Then all the subintegrals will be of the form $P=t^pV$, which substituted in the proposed equation gives,

$$\frac{d^2V}{dv^2}=\frac{d}{dv}\left(v^2\frac{dV}{dv}\right)-2pv\frac{dV}{dv}+p(p-1)V.$$

This equation will be integrable at once, and therefore in finite terms, if we assume $-2p=p(p-1)$,

$$\therefore\ p(p+1)=0.$$

The two roots of this are $p=0$, and $p=1$. The former is of a doubtful character as to whether zero is or is not to be considered to be *not a positive integer;* the latter we see is allowable.

We try the former, rejecting that part of the result which is not in finite terms, and find

$$V = P = A\log\frac{v-1}{v+1}$$

$$= A\log\frac{x-t}{x+t} \quad\text{..........................}\quad (3).$$

To find the second subintegral, or rather the first subintegral corresponding to the second subgeneral integral, we assume $p=-1$.

$$\therefore\ \frac{d^2V}{dv^2} = \frac{d}{dv}\left(v^2\frac{dV}{dv}\right) + 2v\frac{dV}{dv} + 2V.$$

$$\therefore\ V = \frac{Av+B}{v^2-1},$$

and

$$P = t^{-1}V = \frac{Ax+Bt}{x^2-t^2} \quad\text{........................}\quad (4),$$

$$= A(x+t)^{-1} + B(x-t)^{-1},$$

$$\therefore\ u = F\left(\frac{d}{dx}\right)P$$

$$= F(x+t) + f(x-t),$$

which agrees with equation (1).

But as the value of P in (4) is of the dimension (-1) we may by integrating it with regard to x raise it to the dimension zero; in which case the first subintegral will be

$$P = A\log\left(\frac{x-jt}{x+jt}\right)^{-\frac{1}{2}} + B\log\sqrt{x^2-t^2}.$$

73. Change the independent variables of the equation

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2};$$

and let the new variables ξ, η be such that

$$\xi = \log \sqrt{x^2 - t^2}, \text{ and } \eta = \log \left(\frac{x - jt}{x + jt}\right)^{\frac{1}{2}},$$

then

$$\frac{d^2u}{d\xi^2} = \frac{d^2u}{d\eta^2}.$$

Hence the form of the proposed equation is not changed by this change of variables; from which it follows that we are at liberty to write $\log \sqrt{x^2 - t^2}$ for x, and $\log \left(\frac{x - jt}{x + jt}\right)^{\frac{1}{2}}$ for t in any integral of

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2},$$

and the resulting formula will be an integral of the same equation.

74. We shall now consider the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0.$$

We begin with the following proposition respecting this equation and its integral. If in an integral of the proposed equation we write $ax + jby$ for x, and $ay - jbx$ for y, the resulting formula will be an integral of the same equation; a, b being arbitrary constants.

Let $\xi = ax + jby$, and $\eta = ay - jbx$, and let ξ, η be the new independent variables. We find that the result of this change is the following differential equation

$$\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2} = 0.$$

Hence the form of the equation is not affected by this change of the independent variables; and consequently we may write the above values of ξ, η instead of x, y in any known integral of the proposed equation and the resulting formula will also be an integral of it.

On this we may remark that the substitution of $ax + jby$ and $ay - jbx$ for x and y will introduce two germs a, b into the integral; and if the integral in which this substitution is made had been deficient in the number of germs it contained, the integral that results from these substitutions will contain two additional germs, and may possibly now contain the requisite number to render the integral general.

An example will illustrate this.

$$U = \epsilon^x \cos y$$

is manifestly an integral of the proposed equation, and it contains no germs. Make the above substitutions for x and y; then the following is also an integral of the proposed equation, and it contains two germs a, b,

$$U = \epsilon^{ax+jby} \cos (ay - jbx).$$

If into this we introduce the minor germs of x and y we have the following result which is (as we shall presently prove) the general integral of the proposed equation,

$$U = A\epsilon^{ax+jby} \cos (ay - jbx + B) \text{..............} (1).$$

75. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} = 0$.

The general exponential integral is

$$u = A\epsilon^{M(x+iy)} \text{..........................} (1),$$

$$= \phi (x + iy),$$

(in which we are at liberty to write $ax + jby$ for x and $ay - jbx$ for y).

Let $r^2 = x^2 + y^2$, and $\tan \theta = \dfrac{y}{x}$, $\therefore\ x = r \cos \theta$, $y = r \sin \theta$,

$$\begin{aligned}
\therefore\ u &= \phi (r \,.\, \overline{\cos \theta + i \sin \theta}) \\
&= \phi (r\epsilon^{i\theta}) \\
&= \phi (r\epsilon^{i\theta}) + \psi (r\epsilon^{-i\theta}) \\
&= \phi (r^j\epsilon^{i\theta}) \\
&= Ar^{jn}\epsilon^{in\theta} \\
&= (Ar^n + Br^{-n}) (a\epsilon^{n\theta} + b\epsilon^{-n\theta}) \\
&= (Ar^n + Br^{-n}) (a \cos n\theta + b \sin n\theta) \text{..........} (2),
\end{aligned}$$

n being a germ; and A, B, a, b being independent arbitrary constants.

76. Let r and θ be made the independent variables instead of x and y; then the equation of the preceding Article takes the following form,

$$\left(\frac{rd}{dr}\right)^2 u + \frac{d^2u}{d\theta^2} = 0,$$

from which we learn, that we may write in any integral of the equation of the preceding Article r^j for r, and $j\theta$ for θ.

Also θ takes a minor germ, and r a major germ.

If we seek the first subintegral after the manner of Art. 72, we find

$$\text{first subintegral} = A \log \sqrt{x^2+y^2} + B \tan^{-1}\frac{y}{x}.$$

$$\therefore\ u = F\left(\frac{d}{dx}\right) \log \sqrt{x^2+y^2} + f\left(\frac{d}{dx}\right) \tan^{-1}\frac{y}{x} \ldots\ldots\ldots(3).$$

The integrals (1) and (3) are symbolically equivalent.

If $\xi = \log \sqrt{x^2+y^2}$,

$$\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\theta^2} = 0.$$

Hence we may write $\log \sqrt{x^2+y^2}$, $\tan^{-1}\frac{y}{x}$ for x, y in any integral of the equation of the preceding Article, and the resulting formula will be an integral of the same.

77. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} = u$.

The general exponential integral may be presented in the following form,

$$u = A\epsilon^{(M+N)x+(M-N)iy}$$

$$= A\epsilon^{M(x+iy)+N(x-iy)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

subject to the condition $4MN = 1$.

This equation of condition will be satisfied if we assume

$$2M = c\,(\cos m + i \sin m)$$
$$2N = c^{-1}\,(\cos m - i \sin m)$$

in which c and m are real germs, and also independent.

Let $$K = \tfrac{1}{2}\,(x \cos m - y \sin m)$$

and $$I = \tfrac{1}{2}\,(y \cos m + x \sin m)$$

$$\therefore\; u = A\epsilon^{(c+c^{-1})K} \cos \{(c - c^{-1})\, I + B\} \text{..........(2)}.$$

78. If we now consider c a general germ, and assume $2M = c$, and $2N = c^{-1}$, we find the exponential integral in this form,

$$u = A\epsilon^{\frac{1}{2}c(x+iy)+\frac{1}{2}c^{-1}(x-iy)}.$$

This form of the exponential integral agrees, as to its germ c, with equation (1) of Art. 65, from which we learn that the following is the form of the first subintegral P,

$$P = (x + iy)^{-\frac{1}{2}}\,\epsilon^{j\sqrt{(x+iy)(x-iy)}}.$$

This comprehends the two forms, (r^2 being equal to $x^2 + y^2$),

$$P = \{A\,(x + iy)^{-\frac{1}{2}} + B\,(x - iy)^{-\frac{1}{2}}\}\,\epsilon^{jr}.$$

Now $$x + iy = r\,(\cos \theta + i \sin \theta) = r\epsilon^{i\theta}.$$

$$\therefore\; u = \left(\frac{A}{\sqrt{r}}\,\epsilon^{\frac{i\theta}{2}} + \frac{B}{\sqrt{r}}\,\epsilon^{-\frac{i\theta}{2}}\right)\epsilon^{jr}$$

$$= r^{-\frac{1}{2}}\,(A\epsilon^{\frac{i\theta}{2}} + B\epsilon^{-\frac{i\theta}{2}})\,(a\epsilon^{r} + b\epsilon^{-r}) \text{.......(1)},$$

in which A, B, a, b are independent arbitrary constants.

This value is symbolically represented by the following brief equivalent,

$$u = Ar^{-\frac{1}{2}}\,\epsilon^{jr}\,\epsilon^{\frac{i\theta}{2}} \text{.........................(2)}.$$

79. The integral of $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + u = 0$ may be deduced from the preceding Article by changing the algebraic sign of c^{-1} but not that of c.

$$\therefore\; u = A\epsilon^{(c-c^{-1})K} \cos \{(c + c^{-1})\, I + B\},$$

and $$P = (x+iy)^{-\frac{1}{2}}\,\epsilon^{i\sqrt{(x+iy)(iy-x)}}$$

$$= (x+iy)^{-\frac{1}{2}}\,\epsilon^{i\sqrt{(x+iy)(x-iy)}}$$

$$\therefore\ u = Ar^{-\frac{1}{2}}\epsilon^{\frac{1}{2}i\theta}\epsilon^{ir}.$$

80. There are several important equations related to Laplace's equation which are reducible to the following type,

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}.$$

We shall denominate this, when a is a positive quantity, the *standard* equation of this type, for a reason which will be seen presently.

For a small number of particular values of a this equation has been integrated, but for general values of a it has not been integrated.

The value of a admits of reduction in the following manner.

Let $a = 2n + b$, $2n$ being the greatest even integer in a; and let ω be an auxiliary dependent variable such that

$$\frac{d^2\omega}{dt^2} = \frac{d^2\omega}{dx^2} + \frac{b}{x}\frac{d\omega}{dx} \text{ ....(1)}$$

$$= \frac{d}{dx}\left(x\frac{d\omega}{xdx}\right) + b\left(\frac{d\omega}{xdx}\right).$$

Operate on both sides of this equation with $\dfrac{d}{xdx}$,

$$\therefore\ \frac{d^2}{dt^2}\left(\frac{d\omega}{xdx}\right) = \frac{1}{x}\cdot\frac{d^2}{dx^2}\cdot x\left(\frac{d\omega}{xdx}\right) + \frac{b}{x}\frac{d}{dx}\left(\frac{d\omega}{xdx}\right)$$

$$= \frac{d^2}{dx^2}\left(\frac{d\omega}{xdx}\right) + \frac{b+2}{x}\frac{d}{dx}\left(\frac{d\omega}{xdx}\right).$$

On comparing this equation with (1) we perceive that we have here $\dfrac{d\omega}{xdx}$ instead of ω, and $b+2$ instead of b. These changes are simultaneous, and if repeated n times the following would necessarily be the result,

$$\frac{d^2}{dt^2}\left(\frac{d}{xdx}\right)^n\omega = \frac{d^2}{dx^2}\left(\frac{d}{xdx}\right)^n\omega + \frac{b+2n}{x}\frac{d}{dx}\left(\frac{d}{xdx}\right)^n\omega.$$

But $a = 2n + b$,

$$\therefore\ u = \left(\frac{d}{xdx}\right)^n \omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

81. When the integral of the standard equation is known for any positive value of a, the integral for an equal negative value can be deduced from it.

For the equation

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}$$

is immediately reducible to the following form,

$$\frac{d^2u}{dt^2} = x^{-a}\frac{d}{dx} \cdot x^a \frac{du}{dx}.$$

Now operate on this with $\left(x^a \frac{d}{dx}\right)$,

$$\therefore\ \frac{d^2}{dt^2}\left(x^a \frac{du}{dx}\right) = x^a \frac{d}{dx} \cdot x^{-a} \frac{d}{dx}\left(x^a \frac{du}{dx}\right).$$

If now we write ω for $x^a \frac{du}{dx}$ we have

$$\frac{d^2\omega}{dt^2} = x^a \frac{d}{dx} \cdot x^{-a} \frac{d\omega}{dx}$$

$$= \frac{d^2\omega}{dx^2} - \frac{a}{x}\frac{d\omega}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which agrees with the equation in u, with the exception of having $-a$ instead of a. If therefore u the integral of the standard equation be known the integral of (1) will be known from the equation,

$$\omega = x^a \frac{du}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It will therefore be a sufficient solution of the problem of integrating the class of differential equations of the type

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx},$$

if in subsequent articles we confine ourselves to positive values of a.

82. The forms of the following differential equations are all deducible from the general form

$$\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}.$$

$$x\frac{d^2u}{dt^2}=\left(x\frac{d}{dx}+a\right)\frac{du}{dx}$$

$$=x^{-a}\left(x\frac{d}{dx}\right)x^a\cdot\frac{du}{dx},$$

$$\therefore\ \frac{d^2u}{dt^2}=x^{-a}\frac{d}{dx}\cdot x^a\frac{du}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In this write ξ for $(a-1)t$ and η for x^{1-a}.

$$\therefore\ \frac{d^2u}{d\xi^2}=\eta^{\frac{2a}{a-1}}\cdot\frac{d^2u}{d\eta^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This form fails when $a=1$, but in that case the following is the reduced form;

$$x^2\frac{d^2u}{dt^2}=\left(x\frac{d}{dx}\right)^2u\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If in this we write ξ for $\log x$, it takes the following form,

$$\frac{d^2u}{dt^2}=\epsilon^{-2\xi}\frac{d^2u}{d\xi^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

83. The integral of the equation

$$\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}+\frac{bu}{x^2}$$

can also be deduced from that of the standard equation.

Multiply it by x^2,

$$\therefore\ x^2\frac{d^2u}{dt^2}=\left(\frac{xd}{dx}\right)^2u+(a-1)\left(\frac{xd}{dx}\right)u+bu$$

$$=\left(\frac{xd}{dx}+m\right)\left(\frac{xd}{dx}+n\right)u,$$

m, n being the roots of the equation

$$m^2+(1-a)m+b=0.$$

$$\therefore\ x^2\frac{d^2u}{dt^2} = x^{-m}\left(\frac{xd}{dx}\right)x^m . x^{-n}\left(\frac{xd}{dx}\right)x^n u$$

$$\therefore\ \frac{d^2(ux^n)}{dt^2} = x^{-(m-n+1)}\frac{d}{dx} . x^{(m-n+1)}\frac{d(ux^n)}{dx} \text{......(1)},$$

which it will be observed corresponds to the form (1) of the preceding Article, $m-n+1$ taking the place of a, and ux^n of u.

This reduction fails, however, when m and n are equal. In this case $m = \frac{1}{2}(a-1)$,

$$\therefore\ x^2\frac{d^2u}{dt^2} = \left(\frac{xd}{dx}+m\right)^2 u$$

$$= x^{-m}\left(\frac{xd}{dx}\right)^2 . x^m u,$$

$$\therefore\ x^2\frac{d^2(x^m u)}{dt^2} = \left(\frac{xd}{dx}\right)^2 (x^m u) \text{ ....(2)},$$

which agrees with form (3) of the preceding Article.

84. To integrate

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}$$

when a is a positive even integer.

In this case on referring to Art. 80 we find that $b = 0$, and $2n = a$. Hence the auxiliary equation (1) of that article takes the following form

$$\frac{d^2\omega}{dt^2} = \frac{d^2\omega}{dx^2},$$

$$\therefore\ \omega = F(t+x) + f(t-x) = F(t+jx).$$

$$\therefore\ u = \left(\frac{d}{xdx}\right)^n \omega$$

$$= \left(\frac{d}{xdx}\right)^{\frac{a}{2}} F(t+jx).$$

85. To integrate the same equation when $a = 1$, i.e. when the proposed equation is

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} \dots$$

$$\therefore\ x^2\frac{d^2u}{dt^2} = \left(x\frac{d}{dx}\right)^2 u.$$

Now only t can take a minor germ; but this equation is homogeneous on the supposition that t and x are of equal dimensions. Hence $v = \frac{t}{x}$ and V, which is a function of v, are of zero dimensions. We may therefore assume $P = x^p V$ to represent any one of the members of the family of subintegrals. This being substituted in the proposed equation gives the following equation for the determination of V in finite terms;

$$\frac{d^2V}{dv^2} = \frac{d}{dv}\left(v^2\frac{dV}{dv}\right) - (2p+1)\,v\frac{dV}{dv} + p^2V.$$

It is evident that this equation will be integrable if

$$-(2p+1) = p^2,$$

$$\therefore\ 0 = (p+1)^2.$$

We have therefore to deal with a case of equal roots. One integration gives the following result,

$$\frac{dV}{dv} = v^2\frac{dV}{dv} + vV + B.$$

We have now to introduce suppositions, since the form of the integral of this equation will turn upon the relation between t and x.

1. If t^2 is less than x^2, v^2 is less than unity, then the equation to be integrated is

$$(1 - v^2)\frac{dV}{dv} - vV = B.$$

$$\therefore\ V = \frac{A}{\sqrt{1 - v^2}} + \frac{B\sin^{-1}v}{\sqrt{1 - v^2}}.$$

Hence the first subintegral, corresponding to the two subgeneral integrals, is

$$\therefore\ P = x^{-1}\,V = \frac{A}{\sqrt{x^2 - t^2}} + \frac{B\sin^{-1}\dfrac{t}{x}}{\sqrt{x^2 - t^2}} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But this subintegral can be raised to zero dimensions by integrating it with respect to t, for $(x^2 - t^2)^{-\frac{1}{2}} = \dfrac{d}{dt} \cdot \sin^{-1}\dfrac{t}{x}$ (see Art. 25). We may therefore take the following as the first subintegral form of zero dimensions,

$$P = A\left(\sin^{-1}\frac{t}{x}\right) + B\left(\sin^{-1}\frac{t}{x}\right)^2 + B\,(\log mx)^2,$$

in which m is an extemporized major germ.

2. Again, let us now take the case when t^2 is greater than x^2, and therefore v^2 greater than unity.

The equation to be integrated is in this case,

$$(v^2 - 1)\frac{dV}{dv} + vV = B.$$

$$\therefore\ \sqrt{v^2 - 1}\,.\,V = A + B\log(\sqrt{v+1} + \sqrt{v-1}),$$

$$\therefore\ P = (t^2 - x^2)^{-\frac{1}{2}}\left\{A + B\log\left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right)\right\} \ldots (2).$$

This integral can be raised to zero dimensions by integrating it with respect to t, for

$$(t^2 - x^2)^{-\frac{1}{2}} = \frac{d}{dt} \cdot \log\left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right).$$

As only the variable t takes a minor germ, the family of subintegrals can be obtained from (1) and (2) by differentiating or integrating with respect to t.

The following is therefore the general integral required in this Article,

$$u = F\left(\frac{d}{dt}\right)(x^2 - t^2)^{-\frac{1}{2}} + f\left(\frac{d}{dt}\right)\frac{\sin^{-1}\dfrac{t}{x}}{\sqrt{x^2 - t^2}},\ \text{if } x^2 > t^2,$$

or $$u = F\left(\frac{d}{dt}\right)(t^2 - x^2)^{-\frac{1}{2}} + f\left(\frac{d}{dt}\right)\log\left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right),$$

if $t^2 > x^2$.

86. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}$, when a is a positive odd integer.

Referring to Art. 80 we find that $b=1$ in this case, and $n=\frac{1}{2}(a-1)$. Hence the auxiliary equation of that Article takes the following form,

$$\frac{d^2\omega}{dt^2}=\frac{d^2\omega}{dx^2}+\frac{1}{x}\frac{d\omega}{dx},$$

which is the form integrated in the preceding Article;

$$\therefore\ u=\left(\frac{d}{xdx}\right)^{\frac{1}{2}(a-1)}\omega.$$

87. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}$ when a is not an integer.

The differential equation for the determination of V in this case is

$$\frac{d^2V}{dv^2}=\frac{d}{dv}\left(v^2\frac{dV}{dv}\right)-(2p+a)\,v\frac{dV}{dv}+(p^2-p-ap)\,V,$$

and that this may be integrable so as to give V in a finite form we must have $-(2p+a)=p^2-p-ap$;

$$\therefore\ (p+1)(p+a)=0.$$

In this example therefore the two values of p are not equal; and one part of each subintegral will correspond to $p=-1$, and the other to $p=-a$.

1. Let $p=-1$.

Integrating the above on this supposition we find,

$$\frac{dV}{dv}-v^2\frac{dV}{dv}-(2-a)\,vV=B.$$

Multiply this by $(1-v^2)^{-\frac{a}{2}}$ and integrate;

$$\therefore\ (1-v^2)^{1-\frac{a}{2}}V=A+B\int(1-v^2)^{-\frac{a}{2}}dv.$$

We reject the last term, being not integrable in finite terms, and we only require one integral form for this value of p;

$$\therefore\ P = x^{-1}V = Ax^{1-a}(x^2-t^2)^{\frac{a}{2}-1} \text{ .... } (1).$$

2. Let $p = -a$.

Integrating the above equation on this supposition we find

$$(1-v^2)^{\frac{a}{2}}V = A'\int(1-v^2)\,dv + B.$$

We reject the former term of this because it does not give an integral in finite terms and we require only one integral form for this value of p;

$$\therefore\ P = x^{-a}V = B(x^2-t^2)^{-\frac{a}{2}} \text{ .... } (2).$$

Gathering the two parts of P together we have the following complete value of the first subintegral,

$$P = Ax^{1-a}(x^2-t^2)^{\frac{a}{2}-1} + B(x^2-t^2)^{-\frac{a}{2}}.$$

The other members of the subintegral family are to be derived from this equation by differentiation with $\frac{d}{dt}$.

If t^2 be greater than x^2, we may write (t^2-x^2) for (x^2-t^2) in this subintegral.

It will be noticed also that the above subintegrals contain only even powers of t. Subintegrals containing only odd powers of t will be obtained from the above by differentiating once with $\frac{d}{dt}$. We may however pass from P to the general integral which (as only t can take a minor germ) will be (Art. 19)

$$\therefore\ u = F\left(\frac{d}{dt}\right).x^{1-a}(x^2-t^2)^{\frac{a}{2}-1} + f\left(\frac{d}{dt}\right)(x^2-t^2)^{-\frac{a}{2}}.$$

88. To find an integral of the equation

$$\frac{d^2u}{dt^2} + \left(\cos x.\frac{d}{dx}\right)^2 u + n(n+1)\cos^2 x.u = 0.$$

Only t takes a minor germ; and therefore, changing the dependent variable, we may assume $u = e^{mt} X$, in which m is a disposable constant.

$$\therefore \left(\cos x . \frac{d}{dx}\right)^2 X + \{n(n+1)\cos^2 x + m^2\} X = 0.$$

An equation of one variable only, of which a particular integral may be found by assuming $X = \cos^l x$, l being a disposable constant.

$$\therefore l^2 \sin^2 x - l \cos^2 x + n(n+1)\cos^2 x + m^2 = 0;$$

$$\therefore (l^2 + m^2) + (n^2 + n - l^2 - l)\cos^2 x = 0;$$

$$\therefore l^2 + m^2 = 0, \text{ and } n^2 + n = l^2 + l;$$

$$\therefore l = n \text{ or } -(n+1), \text{ and } m = il = in \text{ or } -i(n+1);$$

$$\therefore u = Ae^{int}\cos^n x + Be^{-i(n+1)t}\sec^{n+1} x.$$

We have introduced this example here chiefly for the following reason, and it will be hereafter referred to.

The integral of the equation does not depend directly upon the given value of n, but upon the value of the product $n(n+1)$. Now this product will remain unchanged if we write $-(n+1)$ for n; and consequently the two terms of the above integral belong to it of necessity.

The four following Articles are not specially connected with Laplace's Equation, and therefore do not properly form a part of the present Chapter; but are here introduced as illustrations of the principles laid down in Art. 33 respecting quasi-minor germs.

89. Equations are occasionally met with which are of the following type,

$$\varpi'\left(ax + by, \frac{d}{dx}, \frac{d}{dy}\right) u = 0.$$

We may simplify this form by writing x, y for ax, by. This change of the independent variables will reduce this equation to a form which we may represent by

$$\varpi\left(x + y, \frac{d}{dx}, \frac{d}{dy}\right) u = 0 \quad\text{.................. (1),}$$

and this is the equation which we shall now shew how to reduce to a more convenient form for integration; our object being to obtain an equation in which one of the independent variables shall appear only as a differential symbol of operation (see Art. 18).

In the equation as now before us, though neither of the independent variables can take a minor germ, they can take a quasi-minor germ g; for the equation (1) is in no way affected when $x+g$ is written for x and $y-g$ for y simultaneously.

Hence the general integral of (1) must be of the following form,

$$\begin{aligned}
u &= F(x+g, y-g)\\
&= F(x, y) + \frac{g}{1}\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) + \frac{g^2}{1.2}\left(\frac{d}{dx}-\frac{d}{dy}\right)^2 F(x, y)+\dots\\
&= AF(x, y) + B\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) + C\left(\frac{d}{dx}-\frac{d}{dy}\right)^2 F(x, y)+\dots\\
&= \phi\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) \quad\dots\dots\dots\dots (2).
\end{aligned}$$

And $F(x, y)$ being the first subintegral, all the other subintegrals are deducible from it by successive differentiations with $\left(\frac{d}{dx}-\frac{d}{dy}\right)$.

Now the relation between $\left(\frac{d}{dx}-\frac{d}{dy}\right)$ and $(x+y)$ is such that $\left(\frac{d}{dx}-\frac{d}{dy}\right)(x+y)=0$, and therefore in reference to the compound operation $\left(\frac{d}{dx}-\frac{d}{dy}\right)$ the quantity $(x+y)$ is constant. This suggests the following assumption of new independent variables ξ, η.

Let $\xi = ax - by$, and $\eta = x + y$;

$$\therefore\ (a+b)\frac{d}{d\xi} = \frac{d}{dx} - \frac{d}{dy};$$

and therefore ξ and η are independent variables which satisfy the above conditions.

Also $\frac{d}{dx} = \frac{d}{d\eta} + a\frac{d}{d\xi}$, and $\frac{d}{dy} = \frac{d}{d\eta} - b\frac{d}{d\xi}$,

and the equation becomes

$$\varpi\left(\eta,\ \frac{d}{d\eta} + a\frac{d}{d\xi},\ \frac{d}{d\eta} - b\frac{d}{d\xi}\right)u = 0 \ \ldots\ldots\ldots\ldots\ldots \ (3),$$

from which we see that equation (1) is now reduced to a form in which one of its independent variables (ξ) occurs only as a differential symbol of operation, and will consequently take a minor germ.

The following example is one of historical interest.

90. To integrate $\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{4a}{t+x}\,\frac{du}{dx}$.

We assume $\xi = t - x$, and $\eta = t + x$;

$$\therefore\ \frac{d^2u}{d\xi\, d\eta} = \frac{a}{\eta}\left(\frac{du}{d\eta} - \frac{du}{d\xi}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

We may write $m\xi$, $m\eta$ for ξ, η in this equation without affecting it; hence ξ, η take a major germ; and ξ takes also a minor germ.

This equation (1) has already been integrated in Art. 70;

$$\therefore\ u = A\,(jm\eta - a)^{-a}\epsilon^{jm\xi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$= A\,(m\eta - a)^{-a}\epsilon^{m\xi} + B\,(m\eta + a)^{-a}\epsilon^{-m\xi}.$$

Also assuming $\eta^p V$ for the first subintegral, where V is a function of v, and $v = \frac{\xi}{\eta}$, we find $p = -1$, and the first subintegral

$$P = V\eta^{-1} = \frac{\xi^{a-1}}{\eta^a}\,\epsilon^{\frac{a\xi}{\eta}}\left(A + B\int \epsilon^{-av} v^{-a} dv\right) \ldots\ldots\ (3),$$

$$u = F\left(\frac{d}{d\xi}\right)(P).$$

This integral will be in a finite form only when a is a negative integer. When a is not negative the last term must be rejected.

7
M.

which will be integrable in finite terms if the coefficients o last two terms are equal. This gives the following equati the determination of the two values of p corresponding t two subgeneral integrals:

$$p^2 + (1+a+b)p + (1+a+b+c) = 0.$$

Represent the roots of this equation by $m+jn$; sum $=2m$;

$$\therefore 2m = -(1+a+b);$$

also the equation in V being integrated once gives the lowing:

$$\frac{dV}{dv} + (b-a)V = v^2\frac{dV}{dv} - 2jnvV + B.$$

We omit B as not leading to a finite integral, and because each value of p, i.e. each sign of jn is required furnish only a single integral form of V.

Omitting B and integrating, we find

$$V = \left(\frac{1+v}{1-v}\right)^{\frac{a-b}{2}}(1-v^2)^{jn};$$

$$\therefore P = \eta^p V = \eta^{m+jn}\left(\frac{\eta+\xi}{\eta-\xi}\right)^{\frac{a-b}{2}}\left(1-\frac{\xi^2}{\eta^2}\right)^{jn}$$

$$= \eta^m\left(\frac{\eta+\xi}{\eta-\xi}\right)^{\frac{a-b}{2}}\{A\eta^{-n}(\eta^2-\xi^2)^n + B\eta^n(\eta^2-\xi^2)^{-n}\}\ldots\ldots(2)$$

Also

$$u = F\left(\frac{d}{d\xi}\right)P.$$

P may be expressed in terms of x and y as follows:

$$P = \left(\frac{x}{y}\right)^{\frac{a-b}{2}}(x+y)^m\left\{A\left(\frac{xy}{x+y}\right)^n + B\left(\frac{xy}{x+y}\right)^{-n}\right\}\ldots\ldots\ldots(3$$

and in this case

$$u = F\left(\frac{d}{dx}-\frac{d}{dy}\right)P.$$

93. The preceding Article fails if the roots be equa which case $n=0$, and $m=-\frac{1}{2}(1+a+b)$.

91. An inspection of the integral just found shews that the case of $a=1$ is peculiar, for then it takes the following form,

$$P=\eta^{-1}\epsilon^{\frac{\xi}{\eta}}\left(A+B\int\epsilon^{-v}\frac{dv}{v}\right);$$

omitting the last term as not being a finite form, we have

$$P=A\eta^{-1}\epsilon^{\frac{\xi}{\eta}}.$$

Let this be differentiated with $\left(\frac{d}{d\xi}\right)^n$, or (more generally still) with $\phi\left(\frac{d}{d\xi}\right)$;

$$\therefore\ u=\phi\left(\frac{d}{d\xi}\right)P=A\eta^{-1}\epsilon^{\frac{\xi}{\eta}}\phi\left(\frac{1}{\eta}\right)$$

$$=\phi(t+x)\,.\,\epsilon^{\frac{t-x}{t+x}}.$$

92. To integrate the equation

$$(x+y)^2\frac{d^2u}{dx^2}+(x+y)\left(a\frac{du}{dx}+b\frac{du}{dy}\right)+cu=0.$$

We here assume $x=\eta+\xi$, and $y=\eta-\xi$;

$$\therefore\ \frac{d^2u}{d\xi^2}=\frac{d^2u}{d\eta^2}+\frac{a-b}{\eta}\frac{du}{d\xi}+\frac{a+b}{\eta}\frac{du}{d\eta}+\frac{cu}{\eta^2}\ \ldots\ldots\ldots\ldots\ (1).$$

If $a=b$ this form of equation has been dealt with in Art. 83; but if a and b are unequal we may proceed in the following manner.

The variable ξ takes a minor germ; and also we may write $m\xi$, $m\eta$ for ξ, η without affecting it.

We may therefore assume $v=\frac{\xi}{\eta}$ and $V=$ a function of v; and the general type of subintegrals is $P=\eta^pV$; and then the following will be found to be the differential equation for the determination of V:

$$\frac{d^2V}{dv^2}+(b-a)\frac{dV}{dv}=\frac{d}{dv}\left(v^2\frac{dV}{dv}\right)-(2p+a+b+1)\left(v\frac{dV}{dv}\right)$$

$$+(p^2+ap+bp-p+c)V,$$

which will be integrable in finite terms if the coefficients of the last two terms are equal. This gives the following equation for the determination of the two values of p corresponding to the two subgeneral integrals:

$$p^2 + (1+a+b)p + (1+a+b+c) = 0.$$

Represent the roots of this equation by $m + jn$; their sum $= 2m$;

$$\therefore\ 2m = -(1+a+b);$$

also the equation in V being integrated once gives the following:

$$\frac{dV}{dv} + (b-a)V = v^2\frac{dV}{dv} - 2jnvV + B.$$

We omit B as not leading to a finite integral, and also because each value of p, i.e. each sign of jn is required to furnish only a single integral form of V.

Omitting B and integrating, we find

$$V = \left(\frac{1+v}{1-v}\right)^{\frac{a-b}{2}}(1-v^2)^{jn};$$

$$\therefore\ P = \eta^p V = \eta^{m+jn}\left(\frac{\eta+\xi}{\eta-\xi}\right)^{\frac{a-b}{2}}\left(1-\frac{\xi^2}{\eta^2}\right)^{jn}$$

$$= \eta^m\left(\frac{\eta+\xi}{\eta-\xi}\right)^{\frac{a-b}{2}}\{A\eta^{-n}(\eta^2-\xi^2)^n + B\eta^n(\eta^2-\xi^2)^{-n}\}\ldots\ldots(2).$$

Also $\quad u = F\left(\frac{d}{d\xi}\right)P.$

P may be expressed in terms of x and y as follows:

$$P = \left(\frac{x}{y}\right)^{\frac{a-b}{2}}(x+y)^m\left\{A\left(\frac{xy}{x+y}\right)^n + B\left(\frac{xy}{x+y}\right)^{-n}\right\}\ldots\ldots\ldots(3),$$

and in this case

$$u = F\left(\frac{d}{dx} - \frac{d}{dy}\right)P.$$

93. The preceding Article fails if the roots be equal, in which case $n = 0$, and $m = -\frac{1}{2}(1+a+b)$.

Let $Q = \log \dfrac{xy}{x+y}$; we reduce equation (3) as follows; (= means equivalence);

$$A\left(\frac{xy}{x+y}\right)^{n} + B\left(\frac{xy}{x+y}\right)^{-n} = A\epsilon^{nQ} + B\epsilon^{-nQ}$$

$$= A\left(\epsilon^{nQ} + \epsilon^{-nQ}\right) + \frac{B}{n}\left(\epsilon^{nQ} - \epsilon^{-nQ}\right)$$

$$= A + BQ, \text{ when } n = 0.$$

For equal roots therefore

$$P = \left(\frac{x}{y}\right)^{\frac{a-b}{2}} (x+y)^{-\frac{1+a+b}{2}} \left(A + B \log \frac{xy}{x+y}\right) \text{..........} (4).$$

There still remains the case of imaginary roots, which will be represented by writing in for jn;

$$\therefore \quad P \left(\frac{x}{y}\right)^{\frac{a-b}{2}} (x+y)^{-\frac{1+a+b}{2}} \left\{A\left(\frac{xy}{x+y}\right)^{in} + B\left(\frac{xy}{x+y}\right)^{-in}\right\}$$

$$= \text{.......................} \left\{A \cos nQ + B \sin nQ\right\} \text{......} (5).$$

CHAPTER VII.

EQUATIONS OF THREE INDEPENDENT VARIABLES.

Coefficients constant.

94. ALL the independent variables of equations of this class take minor germs; and therefore a general integral of any such equation will be expressible in an infinite series, every term of which contains only positive integer powers of the variables.

As a general rule the more independent variables are contained in a proposed linear differential equation the more independent major germs may there possibly be; but this is not necessarily the case always. A major germ may perchance belong to only *one*, or to *two* only, or to *all* the independent variables; and thus an individual independent variable may be under the influence of so many as there are different major germs. When the major germs have been introduced into the general exponential integral (Art. 34), we can then eliminate them one by one in any order that shall be found most convenient.

It will be found that the final result of the elimination of all the germs will sometimes depend upon the manner in which major germs were introduced into the original exponential integral; for sometimes they may be introduced in more ways than one. And thus we may obtain in more forms than one a general integral of the proposed equation free from major germs.

95. To integrate the class of equations represented by

$$\varpi\left(\frac{d}{dx},\ \frac{d}{dy}\right)u=\frac{du}{dt}.$$

In this class of equations x, y, t take independent minor germs, and therefore the general integral u is completely expressible in a series containing only positive integer powers of these variables. We may, therefore, assume

$$u = P + Q\frac{t}{1} + R\frac{t^2}{2!} + S\frac{t^3}{3!} + \dots$$

in which P, Q, R, ... are series containing only positive integer powers of x and y, the general type of them all being the following,

$$P = A + \left(B\frac{x}{1} + B'\frac{y}{1}\right) + \left\{C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

Let the above series for u be substituted in the proposed equation;

$$\therefore\ u = \left(1 + \frac{t\varpi}{1} + \frac{t^2\varpi^2}{2!} + \frac{t^3\varpi^3}{3!} + \dots\right)P,$$

in which ϖ is used, for brevity, to represent $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$.

Now in the series which P represents the coefficients are, every one of them, absolutely independent and arbitrary. They are therefore the coefficients of the family of subintegrals of which u is constituted. We may therefore replace them with two independent general germs M, N in the usual manner.

$$\therefore\ P = A\left\{1 + \frac{Mx + Ny}{1} + \frac{(Mx + Ny)^2}{1.2} + \dots\right\}$$

$$= Ae^{Mx+Ny};$$

$$\therefore\ u = \left(1 + \frac{t\varpi}{1} + \frac{t^2\varpi^2}{1.2} + \dots\right).Ae^{Mx+Ny}$$

$$= Ae^{Mx+Ny}\left\{1 + \frac{t.\varpi(M, N)}{1} + \frac{\overline{t.\varpi(M, N)}^2}{1.2} + \dots\right\}$$

$$= Ae^{Mx+Ny+\varpi(M, N)t} \dots\dots\dots\dots (1),$$

which is the usual form of the general exponential integral. That form of the general integral is therefore proved to hold good for all equations of three independent variables of the class proposed in this Article.

We have supposed $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$ not resolvable into equal factorials, for such a case requires a somewhat different treatment. See Art. 69.

96. To integrate the class of equations represented by

$$\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)u=\frac{d^2u}{dt^2},$$

the operative symbol $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$ being supposed to be not resolvable into any factors that are equal.

Following the method of the preceding Article we find in this case,

$$u=\left(1+\frac{t^2\varpi}{2!}+\frac{t^4\varpi^2}{4!}+\ldots\right)P+\left(\frac{t}{1}+\frac{t^3\varpi}{3!}+\frac{t^5\varpi^2}{5!}+\ldots\right)Q,$$

P representing the same series as before, and Q being an independent series of precisely the same form.

The two terms of which u consists are the subgeneral integrals, one of them containing only even, and the other only odd powers of t; and we notice that the *form* of the latter subgeneral integral may be deduced from the former by differentiating with $\frac{d}{dt}$ once.

Now for the same reason as in the preceding Article we may assume, as was there proved, that

$$P=\epsilon^{Mx+Ny}.$$

Let $$\varpi(M, N)=L^2.$$

$$\therefore \text{ first subgeneral integral}=\left(1+\frac{t^2}{2!}\varpi+\frac{t^4}{4!}\varpi^2+\ldots\right)\epsilon^{Mx+Ny}$$

$$=\epsilon^{Mx+Ny}\left(1+\frac{t^2}{2!}L^2+\frac{t^4}{4!}L^4+\ldots\right)$$

$$=A\epsilon^{Mx+Ny}\left(\epsilon^{Lt}+\epsilon^{-Lt}\right);$$

$\therefore$ form of second subgeneral integral is

$$B\epsilon^{Mx+Ny}\left(\epsilon^{Lt}-\epsilon^{-Lt}\right).$$

Both of these terms are comprehended in the one form

$$A\epsilon^{Mx+Ny} \,.\, \epsilon^{jLt}.$$

Hence both the subgeneral integrals may be included in the single form

$$u = A\epsilon^{Mx+Ny+jLt}, \text{ subject to } L^2 = \varpi\,(M, N).$$

Hence the exponential integral of Art. (30) is general and complete for all linear differential equations that belong to the class proposed in this Article.

The existence of the two independent subgeneral integrals in this one expression for u is secured and indicated by the symbol j, which always carries with it the double sign $\pm$.

97. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{du}{dx} + \dfrac{du}{dy}$.

Let the independent variables x, y be changed, the new variables ξ, η being such that $x = \xi + \eta$, and $y = \xi + m\eta$.

$$\therefore\ \frac{du}{dx} + \frac{du}{dy} = \frac{du}{d\xi}, \text{ and } \frac{d^2u}{dt^2} = \frac{du}{d\xi} \ldots\ldots\ldots\ldots\ldots (1).$$

By this change of variables the proposed equation has become an equation of only two independent variables t, ξ; and therefore the remaining variable $\eta = \dfrac{x-y}{1-m}$ is to take the places of the arbitrary constants in the integration of (1).

The integral of (1) we find in Art. 60 to be

$$\therefore\ u = A\epsilon^{M^2\xi+Mt} = \phi\,(x-y) \,.\, \epsilon^{M^2\xi+Mt} \ldots\ldots\ldots\ldots (2).$$

This is the general exponential integral; and M is a general germ.

By the same Article we have the following form of the first subintegral,

$$P = A\xi^{-\frac{1}{2}}\,\epsilon^{-\frac{t^2}{4\xi}} = \frac{\phi\,(x-y)}{\sqrt{mx-y}}\,\epsilon^{\frac{1}{4}\frac{(m-1)\,t^2}{mx-y}} \ldots\ldots\ldots\ldots (3).$$

The value of P contains only even powers of t, and therefore it gives us only one of the subgeneral integrals. But the other subgeneral integral will be obtained from this by differentiating with $\frac{d}{dt}$ (Art. 96). Hence both subgeneral integrals are contained in the following,

$$u = F\left(\frac{d}{dt}\right) \cdot \frac{\phi(x-y)}{\sqrt{mx-y}} \cdot e^{\frac{1}{4}\frac{(m-1)t^2}{mx-y}}.$$

In this m is arbitrary and may be put equal to (-1).

98. To integrate $\frac{du}{dt} = \left(\frac{d}{dx} + \frac{d}{dy}\right)^2 u$.

Proceeding exactly as in the preceding Article we find

$$\frac{du}{dt} = \frac{d^2u}{d\xi^2}.$$

$$\therefore\ u = \phi(x-y) \,.\, e^{M^2t + M\xi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and
$$P = At^{-\frac{1}{2}} e^{-\frac{\xi^2}{4t}} = \frac{\phi(x-y)}{\sqrt{t}} \,.\, e^{-\frac{(mx-y)^2}{4(m-1)^2 t}};$$

$$\therefore\ u = F\left(\frac{d}{d\xi}\right) . \, t^{-\frac{1}{2}} e^{-\frac{\xi^2}{4t}} . \, \phi(x-y) \ldots\ldots\ldots\ldots (2).$$

99. To integrate $\frac{d^2u}{dt^2} = \left(\frac{d}{dx} + \frac{d}{dy}\right)^2 u$.

This may be resolved into the two following independent simple equations,

$$\frac{du}{dt} = \pm \left(\frac{du}{dx} + \frac{du}{dy}\right),$$

which can be integrated in the usual manner.

If we proceed with the proposed equation after the method of the two preceding Articles we find $\frac{d^2u}{dt^2} = \frac{d^2u}{d\xi^2}$, which has been integrated in Art. 72, whence we shall obtain the integral in its various forms; but arbitrary functions of $x - y$ will have to be written instead of the arbitrary constants contained in them.

100. To integrate $\dfrac{d^2u}{dt^2} + \left(\dfrac{d}{dx} + \dfrac{d}{dy}\right)^2 u = 0$.

By the same method the integral of this equation will be obtained from Art. 75.

101. To integrate $\dfrac{du}{dt} = \dfrac{d^2u}{dx\,dy}$.

The general exponential integral of this equation takes the following form,

$$u = A\epsilon^{Mx+Ny+MNt} \dots\dots\dots\dots\dots\dots (1)$$

$$= A\epsilon^{Mx} . \epsilon^{N(y+Mt)}$$

$$= \epsilon^{Mx} \phi(y + Mt) \dots\dots\dots\dots\dots\dots (2).$$

Similarly $$u = \epsilon^{Ny} \psi(x + Nt) \dots\dots\dots\dots\dots\dots (3),$$

the last two being the results of the elimination of one germ only.

To eliminate both the germs from (1) the simplest method will be to change the forms of the germs M, N by assuming $M = m + n$, and $N = m - n$; m, n being two independent germs.

$$\therefore\ u = A\epsilon^{m(x+y)+m^2t} . \epsilon^{n(x-y)+n^2(-t)}.$$

Both m and n may be eliminated by Art. 64; and the chief subintegral is

$$P = t^{-\frac{1}{2}} \epsilon^{-\frac{(x+y)^2}{4t}} . t^{-\frac{1}{2}} \epsilon^{\frac{(x-y)^2}{4t}} = t^{-1} \epsilon^{-\frac{xy}{t}}.$$

This form of P is such that we can at once obtain from it the form of the general integral u.

For $$u = F\left(\frac{d}{dx}\right) P + f\left(\frac{d}{dy}\right) P$$

$$= \frac{1}{t} \left\{F\left(\frac{y}{t}\right) + f\left(\frac{x}{t}\right)\right\} \epsilon^{-\frac{xy}{t}} \dots\dots\dots\dots (4).$$

102. The integral of the equation

$$\frac{du}{dt} = \frac{d^2u}{d\xi^2} + (a + b) \frac{d^2u}{d\xi\, d\eta} + ab \frac{d^2u}{d\eta^2}$$

may be deduced from the preceding Article by assuming as in Art. 67,

$$x=\frac{\eta-b\xi}{a-b},\ \text{and}\ \ y=-\frac{\eta-a\xi}{a-b};$$

$$\therefore\ u=\frac{1}{t}\left\{F\left(\frac{\eta-a\xi}{t}\right)+f\left(\frac{\eta-b\xi}{t}\right)\right\}e^{\frac{(\eta-a\xi)(\eta-b\xi)}{(a-b)^2t}}.$$

103. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx\,dy}$.

The general exponential integral is

$$u=A\epsilon^{Mx+Ny+jLt}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

subject to the condition $L^2=MN$.

We may eliminate the germs, and obtain the first subintegral in the following manner,

$$u=A\epsilon^{Mx}\,.\,\epsilon^{Ny+\sqrt{N}(jt\sqrt{M})}.$$

By Art. 64 this gives the following subintegral by the elimination of N,

$$P=A\epsilon^{Mx}.\,y^{-\frac{1}{2}}\,e^{-\frac{Mt^2}{4y}};$$

$$\therefore\ P=Ay^{-\frac{1}{2}}.\,\epsilon^{M\left(x-\frac{t^2}{4y}\right)}$$

$$=y^{-\frac{1}{2}}\,\phi\left(x-\frac{t^2}{4y}\right).$$

This contains only even powers of t corresponding to one subgeneral integral, and the odd powers which are contained in the other will be contained in $\frac{dP}{dt}$. Both subgeneral integrals are contained in the following formula,

$$u=F\left(\frac{d}{dt}\right).\,y^{-\frac{1}{2}}\,\phi\left(x-\frac{t^2}{4y}\right).$$

But x and y are interchangeable in the proposed equation, and therefore also in P and u. Hence the complete values of P and u are the following:

$$\therefore\ P = y^{-\frac{1}{2}}\phi\left(\frac{4xy-t^2}{y}\right) + x^{-\frac{1}{2}}\psi\left(\frac{4xy-t^2}{x}\right)$$

$$u = F\left(\frac{d}{dt}\right).y^{-\frac{1}{2}}\phi\left(\frac{4xy-t^2}{y}\right) + f\left(\frac{d}{dt}\right).x^{-\frac{1}{2}}\psi\left(\frac{4xy-t^2}{x}\right)\ldots(2).$$

Again, since $MN = L^2$, we may assume in this case,

$$M = Lc\,(\cos m + i\sin m),$$

and $$N = Lc^{-1}(\cos m - i\sin m);$$

$$\therefore\ u = A\epsilon^{L\{(cx+c^{-1}y)\,(\cos m+i\,(cx-c^{-1}y)\sin m+jt\}},$$

in which L, m and c are independent real germs.

Let $K = (cx + cx^{-1}y)\cos m + jt$, and $I = (cx - c^{-1}y)\sin m$.

$$\therefore\ u = A\epsilon^{L(K+iI)}$$

$$= \phi(K+iI) + \psi(K-iI)\ \ldots\ldots\ldots\ldots\ldots\ (3).$$

104. In the first part of the preceding Article we took no account of the fact that the proposed equation allows us to write cx for x, and $c^{-1}y$ for y quite independently of t. The variables x, y have therefore a special relation, and we have therefore to consider the following form of the general exponential integral of that equation,

$$u = A\epsilon^{L(cx+c^{-1}y+jt)}$$

$$= \phi(cx + c^{-1}y + t) + \psi(cx + c^{-1}y - t)\ \ldots\ldots\ldots\ (4),$$

in which c is a general germ.

We may eliminate c by Art. 64, in the following manner:

$$\therefore\ u = A\epsilon^{jLt}.\,\epsilon^{c(Lx)+c^{-1}(Ly)};$$

$$\therefore\ P = \epsilon^{jLt}.\,x^{-\frac{1}{2}}\,\epsilon^{2L\sqrt{xy}}$$

$$= x^{-\frac{1}{2}}.\,\epsilon^{L(2\sqrt{xy}+\underline{\underline{jt}})}$$

$$= x^{-\frac{1}{2}}\phi\,(2\sqrt{xy} + jt).$$

As x and y are interchangeable, the general integral may be presented in the following form,

$$u = F\left(\frac{d}{dx}\right).\,x^{-\frac{1}{2}}\phi\,(2\sqrt{xy}+jt) + f\left(\frac{d}{dy}\right).\,y^{-\frac{1}{2}}\psi\,(2\sqrt{xy}+jt)\ldots(5).$$

105. To integrate $\frac{d^2u}{dt^2} + \frac{d^2u}{dx\,dy} = 0$.

We have merely to write $-x$ for x, or $-y$ for y in the results of the two preceding Articles. Or we may write it for t.

106. The integral of $\frac{d^2u}{dt^2} = \frac{d^2u}{d\xi^2} + (a+b)\frac{d^2u}{d\xi\,d\eta} + ab\,\frac{d^2u}{d\eta^2}$ may be deduced from Art. 103 by means of the same change of the variable x, y as occurs in Art. 67;

$$\therefore\ x = \frac{\eta - b\xi}{a-b} \text{ and } y = -\frac{\eta - a\xi}{a-b},$$

and

$$u = F\left(\frac{d}{dt}\right) . (a\xi - \eta)^{-\frac{1}{2}}\,\phi\left\{\frac{\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2 + \frac{1}{4}(a-b)^2t^2}{a\xi - \eta}\right\}$$

$$+ f\left(\frac{d}{dt}\right) . (b\xi - \eta)^{-\frac{1}{2}}\,\psi\left\{\frac{\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2 + \frac{1}{4}(a-b)^2t^2}{b\xi - \eta}\right\} \ldots (1).$$

We may derive the following form of u from Art. 104:

$$u = F\left(\frac{d}{d\xi} + a\,\frac{d}{d\eta}\right) . (b\xi - \eta)^{-\frac{1}{2}}\,\phi\,\{(\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2)^{\frac{1}{2}}$$
$$+ \tfrac{1}{2}(a-b)\,it\}$$

$$+ f\left(\frac{d}{d\xi} + b\,\frac{d}{d\eta}\right) . (a\xi - \eta)^{-\frac{1}{2}}\,\psi\,\{(\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2)^{\frac{1}{2}}$$
$$+ \tfrac{1}{2}(a-b)\,it\} \ldots (2).$$

107. Let $a = 1$ and $b = -1$ in the preceding Article; then the integrals of the equation

$$\frac{d^2u}{d\xi^2} = \frac{d^2u}{dt^2} + \frac{d^2u}{d\eta^2}$$

will be of the following forms,

$$u = F\left(\frac{d}{dt}\right) . (\xi - \eta)^{-\frac{1}{2}}\,\phi\left(\frac{\eta^2 - \xi^2 + t^2}{\xi - \eta}\right)$$
$$+ f\left(\frac{d}{dt}\right) . (\xi + \eta)^{-\frac{1}{2}}\,\psi\left(\frac{\eta^2 - \xi^2 + t^2}{\xi + \eta}\right).$$

But t and η are interchangeable, and the two terms of this integral may be represented as one; the following is therefore the form in greater detail,

$$\therefore\ u = F\left(\frac{d}{dt}\right) . (\xi + j\eta)^{-\frac{1}{2}} \phi \frac{t^2 + \eta^2 - \xi^2}{\xi + j\eta}$$

$$+ f\left(\frac{d}{d\eta}\right) . (\xi + jt)^{-\frac{1}{2}} \psi \left(\frac{t^2 + \eta^2 - \xi^2}{\xi + jt}\right) \text{.......} (1).$$

The second form of u will be the following,

$$u = F\left(\frac{d}{d\xi} + j\frac{d}{d\eta}\right) . (\xi + j\eta)^{-\frac{1}{2}} \phi \{(\eta^2 - \xi^2)^{\frac{1}{2}} + it\}$$

$$+ f\left(\frac{d}{d\xi} + j\frac{d}{dt}\right) . (\xi + jt)^{-\frac{1}{2}} \psi \{(t^2 - \xi^2)^{\frac{1}{2}} + i\eta\} \text{......} (2).$$

108. If in the preceding Article we write $i\xi$ for ξ, the integrals of the equation

$$\frac{d^2u}{dt^2} + \frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2} = 0$$

will be

$$u = F\left(\frac{d}{dt}\right) . (\eta + i\xi)^{-\frac{1}{2}} \phi \left(\frac{t^2 + \xi^2 + \eta^2}{\eta + i\xi}\right) \text{............} (1),$$

and

$$u = F\left(\frac{d}{d\eta} + i\frac{d}{d\xi}\right) . (\eta + i\xi)^{-\frac{1}{2}} \phi (\sqrt{\xi^2 + \eta^2} + it) \text{......} (2).$$

It being understood in these results that t, ξ, η are all interchangeable.

109. To integrate $\dfrac{du}{dt} = \dfrac{d^2u}{dx\,dy} + ju$.

This equation takes the form

$$\frac{d . u\epsilon^{-jt}}{dt} = \frac{d^2 . u\epsilon^{-jt}}{dx\,dy},$$

a form which is integrated in Art. 101.

110. To integrate $\frac{d^2u}{dt^2} = \frac{d^2u}{dx\,dy} + u.$

The general exponential integral of this equation is

$$u = A\epsilon^{Mx+Ny+jLt},$$

subject to the conditional equation

$$L^2 = MN + 1 \text{ or } L^2 - 1 = MN.$$

We may here assume

$$M = c(L+1), \text{ and } N = c^{-1}(L-1),$$

c being a general germ;

$$\therefore Mx + Ny + jLt = L(cx + c^{-1}y + jt) + cx - c^{-1}y;$$

$$\therefore u = A\epsilon^{cx-c^{-1}y}\,.\,\epsilon^{L(cx+c^{-1}y+jt)}$$

$$= A\epsilon^{cx-c^{-1}y}\,\phi(cx + c^{-1}y + jt) \,\ldots\ldots\ldots (1).$$

The germ c still remains uneliminated; we shall therefore now shew how to eliminate both the germs (L and c) contained in the exponential integral,

$$u = A\epsilon^{jLt}\,.\,\epsilon^{c(L+1)x+c^{-1}(L-1)y}.$$

Hence by Art. 65, eliminating c, the following is the corresponding first subintegral,

$$P = \epsilon^{jLt}\,.\,x^{-\frac{1}{2}}\,\epsilon^{\pm 2\sqrt{(L^2-1)xy}};$$

$$\therefore P\sqrt{x} = \epsilon^{jLt\pm 2\sqrt{(L^2-1)xy}}.$$

Let now $\xi = 2\sqrt{xy}$, then eliminating L by differentiation of the last equation we find

$$\therefore \frac{d^2(P\sqrt{x})}{dt^2} = \frac{d^2(P\sqrt{x})}{d\xi^2} + (P\sqrt{x}),$$

and by the method of Art. 78 the first subintegral of this equation is

$$= (\xi + jt)^{-\frac{1}{2}}\epsilon^{\pm\sqrt{t^2-\xi^2}}$$

$$= (2\sqrt{xy} + jt)^{-\frac{1}{2}}\,\epsilon^{\pm\sqrt{t^2-4xy}}.$$

Hence the first subintegral of the proposed equation is

$$P = x^{-\frac{1}{2}}(2\sqrt{xy}+jt)^{-\frac{1}{2}}\epsilon^{\pm\sqrt{t^2-4xy}} \;\ldots\ldots\ldots\ldots\ldots\; (2),$$

whence u is known. The algebraic signs j and $\pm$ in this integral are independent; and x, y are interchangeable.

111. In the preceding Article c is a general germ; but if we wish to have the integral which is equivalent to (1) in real germs, we may write

$c(\cos m + i\sin m)$ for c, and $c^{-1}(\cos m - i\sin m)$ for c^{-1},

c and m being now real germs.

112. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{d^2u}{dx\,dy} - u$.

Here the exponential integral is

$$u = A\epsilon^{Mx+Ny+Ljt},$$

subject to the condition

$$L^2 = MN - 1, \quad\text{or}\quad L^2 - i^2 = MN.$$

We now assume

$$M = c(L+i), \quad\text{and}\quad N = c^{-1}(L-i),$$

c being a general germ;

$$\therefore\; u = A\epsilon^{L(cx+c^{-1}y+jt)}\,.\,\epsilon^{i(cx-c^{-1}y)}$$

$$= \phi(cx+c^{-1}y+jt)\,.\,\cos(cx-c^{-1}y+B) \;\ldots\ldots\; (1).$$

Following the method of Art. 110 we have

$$u = A\epsilon^{jLt}\,.\,\epsilon^{c\,L+i)x+c^{-1}(L-i)y};$$

$$\therefore\; P = \epsilon^{jLt}\,.\,x^{-\frac{1}{2}}\epsilon^{\pm 2\sqrt{(L^2+1)xy}};$$

$$\therefore\; \frac{d^2(P\sqrt{x})}{d\xi^4} = \frac{d^2(P\sqrt{x})}{dt^2} + (P\sqrt{x});$$

$$\therefore\; P = x^{-\frac{1}{2}}(2\sqrt{xy}+jt)^{-\frac{1}{2}}\epsilon^{\pm\sqrt{4xy-t^2}} \;\ldots\ldots\ldots\ldots\; (2),$$

whence u is known. As before x, y are interchangeable.

113. In Art. 74 we have shewn that the independent variables of the equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ may be changed without affecting the form of the equation itself. We shall now prove a corresponding property for the more general class of equations included in the following form,

$$\frac{d^2u}{dx^2}\pm\frac{d^2u}{dy^2}=\varpi\left(t,\ \frac{d}{dt}\right)u \text{ .... (1).}$$

Instead of x and y assume two new independent variables ξ, η, such that $\xi=Mx+jmy$, and $\eta=mx\mp jMy$; the disposable constants M, m being subject to the following condition,

$$M^2\pm m^2=1 \text{ ....(2).}$$

Let the integral of (1) be $u=F(x,\ y,\ t)$; and let $W=F(\xi,\ \eta,\ t)$.

$$\therefore \frac{d^2W}{d\xi^2}\pm\frac{d^2W}{d\eta^2}=\varpi\left(t,\ \frac{d}{dt}\right)W.$$

But $$\frac{d^2W}{dx^2}=M^2\frac{d^2W}{d\xi^2}+2Mm\frac{d^2W}{d\xi\,d\eta}+m^2\frac{d^2W}{d\eta^2},$$

and $$\frac{d^2W}{dy^2}=m^2\frac{d^2W}{d\xi^2}\mp 2Mm\frac{d^2W}{d\xi\,d\eta}+M^2\frac{d^2W}{d\eta^2};$$

$$\therefore \frac{d^2W}{dx^2}\pm\frac{d^2W}{dy^2}=\frac{d^2W}{d\xi^2}\pm\frac{d^2W}{d\eta^2}=\varpi\left(t,\ \frac{d}{dt}\right)W.$$

On comparing the last result with the equation (1) we see that W is an integral of (1). And as ξ, η contain a germ that is not contained in $F(x, y, t)$, the integral $F(\xi, \eta, t)$ will contain that germ, and be at least as general as $F(x, y, t)$.

Hence without diminishing the generality (and with a chance of increasing it) we may write $Mx+jmy$ and $mx\mp jMy$ for x and y in any integral of equation (1).

In equation (2) we may substitute $\frac{1}{2}(c+c^{-1})$ for M, and $\frac{i}{2}(c-c^{-1})$ for m when the upper sign of m^2 is used in (2); but when the lower sign is used we must substitute $\frac{j}{2}(c - c^{-1})$ for m, c being a general germ.

The double sign in this Article is regulated by that in equation (1).

114. It will be observed as a property connecting the two sets of independent variables used in the preceding Article, that

$$\xi^2 \pm \eta^2 = x^2 \pm y^2.$$

We may represent either of these quantities by r^2. Hence in passing from the equation in terms of x, y to the equivalent equation in ξ, η, and expressing the results in terms of r and another independent variable, the quantity r will occur in the two resulting forms in the same manner, and be of the same value in both.

115. To integrate $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{du}{dt}$.

The general exponential integral may be written in the following form:

$$u = A\epsilon^{Mx+Ny+(M^2+N^2)t}$$

$$= A\epsilon^{M^2t+Mx}\,\epsilon^{N^2t+Ny}.$$

By Art. 64 the germs M, N may be eliminated, and the following is the general form of the subintegrals,

$$P = t^{-\frac{1}{2}}\,\epsilon^{-\frac{x^2}{4t}}\,.\,t^{-\frac{1}{2}}\epsilon^{-\frac{y^2}{4t}}$$

$$= t^{-1}\,\epsilon^{-\frac{x^2+y^2}{4t}} = t^{-1}\,\epsilon^{\frac{(ix+y)(ix-y)}{4t}}.$$

From this we may find u in the following manner,

$$u = F\left(\frac{d}{dx} - i\frac{d}{dy}\right)P + f\left(\frac{d}{dx} + i\frac{d}{dy}\right)P$$

$$= t^{-1}\left\{F\left(\frac{y+ix}{t}\right) + f\left(\frac{y-ix}{t}\right)\right\}\epsilon^{-\frac{x^2+y^2}{4t}} \quad \text{.........} \quad (1).$$

If now we assume $x = r\sin\theta$, and $y = r\cos\theta$, this integral takes the following form, since

$$y + ix = r(\cos\theta + i\sin\theta) = r\epsilon^{i\theta},$$

$$u = \frac{1}{t}\epsilon^{-\frac{r^2}{4t}}\left\{F\left(\frac{r}{t}\epsilon^{i\theta}\right) + f\left(\frac{r}{t}\epsilon^{-i\theta}\right)\right\}$$

$$= \frac{1}{t}\epsilon^{-\frac{r^2}{4t}}F\left(\frac{r}{t}\epsilon^{i\theta}\right) \text{ .... (2).}$$

116. To integrate

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = \frac{du}{dt}.$$

We might deduce this from the preceding Article by merely writing iy for y; but we shall integrate this equation in an independent manner.

Let ξ, η be a new set of independent variables such that $\xi = x + y$ and $\eta = x - y$;

$$\therefore \frac{d^2u}{d\xi\, d\eta} = \frac{du}{4dt}.$$

This agrees in form with the equation integrated in Art. 101;

$$\therefore u = \frac{1}{t}\left\{F\left(\frac{\eta}{t}\right) + f\left(\frac{\xi}{t}\right)\right\}\epsilon^{-\frac{\xi\eta}{4t}}$$

$$= \frac{1}{t}\epsilon^{\frac{y^2-x^2}{4t}}\left\{F\left(\frac{x-y}{t}\right) + f\left(\frac{x+y}{t}\right)\right\}.$$

117. To integrate

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{dt^2}.$$

This has been integrated in Art. 107, but the following method will serve as an illustration of the variety of ways in which germs may be introduced into the general exponential integral:

$$u = A\epsilon^{(M+N)t+(M-N)x+2\sqrt{MN}.y}$$

$$= A\epsilon^{M(t+x)}\,.\,\epsilon^{N(t-x)+\sqrt{N}(2y\sqrt{M})}.$$

Eliminate N; then the first subintegral is

$$P = \epsilon^{M(t+x)} \,.\, (t-x)^{-\frac{1}{2}}\, \epsilon^{\frac{My^2}{x-t}}$$

$$= (t-x)^{-\frac{1}{2}} \,.\, \epsilon^{M\left(t+x+\frac{y^2}{x-t}\right)}$$

$$= (t-x)^{-\frac{1}{2}}\, F\left(\frac{x^2+y^2-t^2}{x-t}\right).$$

The proposed equation shews that in any integral we may write jt for t. We also notice that x and y are interchangeable. Introducing these properties, we find the following general integral,

$$u = (x+jt)^{-\frac{1}{2}} F\left(\frac{x^2+y^2-t^2}{x+jt}\right) + (y+jt)^{-\frac{1}{2}} f\left(\frac{x^2+y^2-t^2}{y+jt}\right).$$

118. The integral of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

will be deduced from the preceding Article by writing iz for jt;

$$\therefore\ u = (x+iz)^{-\frac{1}{2}} F\left(\frac{x^2+y^2+z^2}{x+iz}\right) + (y+iz)^{-\frac{1}{2}} f\left(\frac{x^2+y^2+z^2}{y+iz}\right).$$

This is a complete general integral of Laplace's equation.

119. By a different distribution of the major germs from that in Art. 117 we may obtain in another form a general integral of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{dt^2}.$$

Let
$$u = A\epsilon^{L\left\{(c+c^{-1})\frac{x}{2} + i(c-c^{-1})\frac{y}{2} + jt\right\}}$$

$$= A\epsilon^{jLt} \,.\, \epsilon^{c(x+iy)\frac{L}{2} + c^{-1}(x-iy)\frac{L}{2}}.$$

Eliminate c by Art. 65; then the following is the first of the subintegrals,

$$P = \epsilon^{jLt} \,.\, (x+iy)^{-\frac{1}{2}}\, \epsilon^{\pm L\sqrt{x^2+y^2}}$$

$$= (x+iy)^{-\frac{1}{2}}\, \epsilon^{L\left(jt\pm\sqrt{x^2+y^2}\right)};$$

$$\therefore\ u = (x+iy)^{-\frac{1}{2}} F\left(jt \pm \sqrt{x^2+y^2}\right).$$

120. If in preceding Article we write iz for jt, we find the integral of the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$$

in the following form,

$$u=(x+iy)^{-\frac{1}{2}}F\left(\sqrt{x^2+y^2}\pm iz\right),$$

which is a form of the general integral of Laplace's equation agreeing with (2) in Art. 108.

121. To extend to three independent variables the property proved in Art. 113 for two.

Our equation is now of the following form,

$$\varpi\left(t,\frac{d}{dt}\right)u=\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\;(1).$$

Let $r^2=x^2+y^2+z^2$; and let ξ, η, ζ be new independent variables such that

$$\xi=ax+a'y+a''z,$$
$$\eta=bx+b'y+b''z,$$
$$\zeta=cx+c'y+c''z.$$

The nine constants in these expressions are disposable; and we are to dispose of them in such a way that when these values of ξ, η, ζ are written for x, y, z in any integral of equation (1) the resulting formula will also be an integral of the same equation.

Let $u=F(x,y,z,t)$ be any integral of equation (1), then is $V=F(\xi,\eta,\zeta,t)$ an integral of the equation

$$\varpi\left(t,\frac{d}{dt}\right)V=\frac{d^2u}{d\xi^2}+\frac{d^2u}{d\eta^2}+\frac{d^2u}{d\zeta^2}\;\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But $$\frac{dV}{dx}=a\frac{dV}{d\xi}+b\frac{dV}{d\eta}+c\frac{dV}{d\zeta}=\left(\frac{ad}{d\xi}+\frac{bd}{d\eta}+\frac{cd}{d\zeta}\right)V;$$

$$\therefore\;\frac{d^2V}{dx^2}=\left(\frac{ad}{d\xi}+\frac{bd}{d\eta}+\frac{cd}{d\zeta}\right)^2V.$$

Similarly $$\frac{d^2V}{dy^2}=\left(\frac{a'd}{d\xi}+\frac{b'd}{d\eta}+\frac{c'd}{d\zeta}\right)^2V,$$

and $$\frac{d^2V}{dz^2}=\left(\frac{a''d}{d\xi}+\frac{b''d}{d\eta}+\frac{c''d}{d\zeta}\right)^2V.$$

Expanding and adding together the right-hand members of these three equations, and assuming the following six relations among the nine disposable constants,

$$1=a^2+a'^2+a''^2=b^2+b'^2+b''^2=c^2+c'^2+c''^2$$

and $$0=ab+a'b'+a''b''=ac+a'c'+a''c''=bc+b'c'+b''c'',$$

we have the following general result,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=\frac{d^2V}{d\xi^2}+\frac{d^2V}{d\eta^2}+\frac{d^2V}{d\zeta^2}$$

$$=\varpi\left(t,\frac{d}{dt}\right)V, \text{ by equat. (2).}$$

Hence V, which is equal to $F(\xi, \eta, \zeta, t)$, is an integral of equation (1) when the above values of ξ, η, ζ are written for x, y, z in $F(x, y, z, t)$, which is equal to u.

122. It will be observed that the nine disposable constants have to satisfy only six conditional equations; and moreover that those six equations are such as prove the following general relation between x, y, z and ξ, η, ζ,

$$x^2+y^2+z^2=\xi^2+\eta^2+\zeta^2.$$

Hence r^2, which is equal to $x^2+y^2+z^2$, does not change in value when we pass from x, y, z to the values represented by ξ, η, ζ.

123. In reference to the equation (1) of Art. 121 we are aware that x, y, z which it contains are not necessarily the co-ordinates of a point P in space, nor is there necessarily any system of co-ordinates to which they are referred; but for convenience in what follows we will suppose them referred to an arbitrary rectangular system of co-ordinates Ox, Oy, Oz, and we will set off upon these axes the values of x, y, z; and suppose P

the point in space of which x, y, z are thus constituted the rectangular co-ordinates;

$$\therefore\ OP^2 = r^2 = x^2 + y^2 + z^2.$$

Now on these same co-ordinate axes set off a, a', a'' as the co-ordinates of a fixed point A; and let b, b', b'' be those of B, and c, c', c'' those of C.

Then if we assume $OA = OB = OC = 1$, these assumptions satisfy the first three of the conditions to be satisfied by the nine disposable constants; and if these lines OA, OB, OC be now supposed to be at right angles to each other, the following equations shew that the constants will then satisfy the remaining three conditions also. For then

$$ab + a'b' + a''b'' = \cos AOB = \cos\frac{\pi}{2} = 0,$$

$$ac + a'c' + a''c'' = \cos AOC = \cos\frac{\pi}{2} = 0,$$

$$bc + b'c' + b''c'' = \cos BOC = \cos\frac{\pi}{2} = 0.$$

Hence, if OA, OB, OC be each equal to unity and mutually at right angles, the six conditional equations are all satisfied; and OA, OB, OC may be taken as an arbitrary system of rectangular co-ordinate axes. We say an *arbitrary* system, because of the nine disposable constants on which the positions of OA, OB, OC depend three are still disposable, and these render the position of this system so far arbitrary.

Now as x, y, z are the co-ordinates of P, and $OP = r$,

$$x = r\cos xOP,\ y = r\cos yOP,\ z = r\cos zOP;$$

$$\therefore\ \cos AOP = a\frac{x}{r} + a'\frac{y}{r} + a''\frac{z}{r} = \frac{\xi}{r};$$

$$\therefore\ \xi = r\cos AOP.$$

Similarly $\qquad \eta = r\cos BOP,$

and $\qquad \zeta = r\cos COP.$

Therefore ξ, η, ζ are the co-ordinates of P referred to the arbitrary system of rectangular co-ordinate axes OA, OB, OC; the position of which in reference to the original fixed system Ox, Oy, Oz depends upon the values arbitrarily assigned to the remaining three still disposable constants; which we may in fact describe as three disposable real germs.

When therefore we have before us an equation of the class comprehended in the equation

$$\varpi\left(t,\ \frac{d}{dt}\right)u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}$$

we are at liberty to substitute, in any integral of it, in the places of x, y, z, the values of ξ, η, ζ in terms of x, y, z given in Art. 121, and the formula produced by such substitution will be an integral of the same equation; and will virtually contain three real germs which were not contained in the original form of the integral. And moreover the value of r^2 will not thereby be affected.

Laplace's Equation, $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$.

124. The following is the simplest form of the general exponential integral,

$$u = A\epsilon^{L(Mx+Ny+iz)},$$

subject to the condition $M^2 + N^2 = 1$.

This integral is equivalent to the following form,

$$u = A\epsilon^{L(Mx+Ny)} \cos(Lz + B).$$

If we now write in this for x, y, z the values of ξ, η, ζ found in Art. 121, we shall have a general *form* which may be thus represented,

$$u = A\epsilon^{L(ax+a'y+a''z)} \cos L\,(bx + b'y + b''z + B),$$

(a, a', a'', b, b', b'' are not here the same as in Art. 121, but at present they are disposable).

Let the cosine be replaced by its exponential equivalent;

$$\therefore\ u = A\epsilon^{L(\overline{a+ib}\,.\,x+\overline{a'+ib'}\,.\,y+\overline{a''+ib''}\cdot z)}.$$

That this may be the general integral, the following condition must be satisfied,

$$0 = (a+ib)^2 + (a'+ib')^2 + (a''+ib'')^2,$$

which separates itself into the two following independent conditions,

$$a^2 + a'^2 + a''^2 = b^2 + b'^2 + b''^2,$$

$$ab + a'b' + a''b'' = 0.$$

Now the presence of the arbitrary germ L permits us, without any loss of generality, to assume

$$a^2 + a'^2 + a''^2 = b^2 + b'^2 + b''^2 = 1:$$

and this assumption being made we find, on reference to Art. 121, that these six disposable constants are identical with the corresponding six in that Article;

$$\therefore\ ax + a'y + a''z = \xi,$$

$$\text{and } bx + b'y + b''z = \eta,$$

and consequently the general exponential integral may be expressed in the following brief form,

$$u = A\epsilon^{L\xi}\cos(L\eta + B) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the above equations of conditions among the six constants a, a', a'', b, b', b'', shew that ξ and η are interchangeable;

$$\therefore\ u = A'\epsilon^{L\eta}\cos(L\xi + B') \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Either of these results may be regarded as a general integral of Laplace's equation; or by addition they may be combined in a single integral.

The six disposable constants are subject to only three conditional equations.

On reference to Art. 123 we perceive that ξ, η are the co-ordinates of P referred to the two lines OA, OB; while x, y, z are the co-ordinates of P referred to the three fixed rectangular axes Ox, Oy, Oz.

Hence ξ, η are the projections of OP (i.e. of r) upon OA, OB respectively; and as the positions of these two rectangular axes are dependent on three arbitrary germs, therefore the values of ξ, η involve those three germs, and consequently the integral

$$u = A\epsilon^{L\eta} \cos (L\xi + B)$$

is a germ integral.

If we assign *particular* values to the three germs we obtain from this a *particular* integral: and as particular values of germs may be infinite in number, we may thus obtain an unlimited number of particular integrals, which we may denominate a new family of subintegrals: and out of them it may be possible by proper management to construct a general integral in finite terms in a form suitable to a physical or geometrical problem which we may have in hand.

125. To integrate $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2z}{dz^2} = 0.$

Assume the following form of the general exponential integral,

$$u = A\epsilon^{m(iy-x)} . \epsilon^{M(iy+x)+\sqrt{M}(2z\sqrt{m})}.$$

Eliminate M; then the form of the resulting subintegral is,

$$P = \epsilon^{m(iy-x)} . (iy+x)^{-\frac{1}{2}} \epsilon^{\frac{-mz^2}{iy+x}}$$

$$= (iy+x)^{-\frac{1}{2}} \epsilon^{m\left(iy-x-\frac{z^2}{iy+x}\right)};$$

$$\therefore\ u = (iy + x)^{-\frac{1}{2}} F\left(iy - x - \frac{z^2}{iy + x}\right)$$

$$= (x + iy)^{-\frac{1}{2}} F\left(\frac{r^2}{x + iy}\right)$$

$$= \frac{1}{r}\left(\frac{r^2}{x + iy}\right)^{\frac{1}{2}} F\left(\frac{r^2}{x + iy}\right)$$

$$= \frac{1}{r} F\left(\frac{r^2}{x + iy}\right), \text{ or, } = \frac{1}{r} F\left(\frac{x + iy}{r^2}\right) \quad\ldots\ldots\ldots\ldots(1)$$

$$= \frac{A}{r}\left(\frac{x + iy}{r^2}\right)^m, \ m \text{ being a germ,}$$

$$= \frac{A\,(x + iy)^m}{r^{2m+1}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This integral contains only even powers of z; consequently this is only one of the subgeneral integral forms. The other subgeneral form will be found from this by differentiating with $\frac{d}{dz}$.

Let us now assume $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ $z = r\sin\theta$;

$$\therefore\ x + iy = r\cos\theta\,(\cos\phi + i\sin\phi) = r\cos\theta\,\epsilon^{i\phi};$$

$$\therefore\ u = \frac{A}{r}\left(\frac{x + iy}{r^2}\right)^m = \frac{A}{r}\left(\frac{\cos\theta}{r}\,.\,\epsilon^{i\phi}\right)^m$$

$$= \frac{1}{r} F\left(\frac{\cos\theta}{r}\,.\,\epsilon^{i\phi}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This subgeneral integral contains two independent arbitrary functions by reason of the double sign of i.

We may obtain another form of the subgeneral integral in the following manner:

126. To integrate $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$.

Assume the following form of the general exponential integral,

$$u = A\epsilon^{N(Mx + my + iz)},$$

the germs M, m being subject to the equation $M^2 + m^2 = 1$;

$$\therefore\ u = A\epsilon^{iNz} . \epsilon^{M(Nx)+m(Ny)}.$$

Eliminate M and m; the following is the corresponding form of the subintegral:

$$P = (x \pm iy)^{-\frac{1}{2}} \epsilon^{N(j\sqrt{x^2+y^2}+iz)};$$

$$\therefore\ u = (x \pm iy)^{-\frac{1}{2}} F(\sqrt{x^2+y^2} + iz) \ldots\ldots\ldots\ldots\ldots (1)$$

$$= A\ (x \pm iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} + iz)^m,$$

m being here an extemporized germ. Also y and z are interchangeable.

But $x \pm iy = r\epsilon^{\pm i\phi} \cos\theta$, and $\sqrt{x^2+y^2} + iz = r\epsilon^{i\theta}$;

$$\therefore\ u = A(r\epsilon^{\pm i\phi} \cos\theta)^{-\frac{1}{2}} (r\epsilon^{i\theta})^m$$

$$= (r\epsilon^{\pm i\phi} \cos\theta)^{-\frac{1}{2}} F(r\epsilon^{i\theta}) \ \ldots\ldots\ldots\ldots\ldots (2).$$

Also $$u = Ar^{m-\frac{1}{2}} \sqrt{\cos\theta . \epsilon^{\pm i\phi}} . \epsilon^{im\theta} \ \ldots\ldots\ldots\ldots (3).$$

The form of this integral differs from that found in the preceding Article, and illustrates the power we have over the form of the general integral, since we can introduce the germs into the general exponential integral in more than one set of different relationships to the independent variables.

Laplace's Functions.

127. Let the independent variables in Laplace's equation be changed from x, y, z to r, θ, ϕ; these being defined by the equations before given, viz.,

$$x = r\cos\theta\cos\phi,\ y = r\cos\theta\sin\phi,\ z = r\sin\theta;$$

$$\therefore\ x^2 + y^2 + z^2 = r^2;$$

and the transformed equation is

$$0 = r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} + \frac{d^2u}{d\theta^2} - \tan\theta . \frac{du}{d\theta} + \sec^2\theta\frac{d^2u}{d\phi^2},$$

or more conveniently in the following form,

$$0 = \cos^2\theta \cdot \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right) u + \left(\cos\theta \cdot \frac{d}{d\theta}\right)^2 u + \frac{d^2u}{d\phi^2} \dots\dots(1).$$

Concerning this form of Laplace's equation we remark that r alone takes a major germ; and ϕ alone takes a minor germ. Also $j\theta$, $j\phi$ may at any time be independently written for θ, ϕ in any integral of it.

Now as mr may be written for r (m being a major germ) in the integral of (1), that integral (Art. 13) may be expanded in a series in powers of m;

$$\therefore\ u = m^p \cdot r^p W_p + m^q \cdot r^q W_q + \dots$$

and the powers of m in this equation may be replaced with independent arbitrary constants, which are in fact the coefficients of the subintegrals of u. The quantities denoted by W are functions, not of r, but of θ and ϕ.

We may therefore take the following as the general representative of the subintegrals of u,

$$P = r^n W.$$

To determine W corresponding to a given value of n we substitute this subintegral in equation (1), and the result is,

$$0 = n(n+1)\cos^2\theta \cdot W + \left(\cos\theta \cdot \frac{d}{d\theta}\right)^2 W + \frac{d^2W}{d\phi^2} = 0 \dots(2).$$

This is the Equation of Laplace's Functions, and from this equation we perceive that r has been divided out, leaving W, the representative of Laplace's Functions, dependent on the value of the product $n(n+1)$, the only remanet of r.

128. *Lemma.* The numerical value of the product $(n+a)(n+b)$ will suffer no change if we write in it $-(n+a+b)$ for n.

Hence $n(n+1)$ remains unchanged in value when $-(n+1)$ is written for n.

This shews that instead of assuming the general form of the subintegrals in the preceding Article to be $P = r^n W$, this will not be a general assumption unless we write $Ar^n + ar^{-n-1}$ for r^n;

$$\therefore\ P = (Ar^n + ar^{-n-1})W$$

is the general form of the subintegrals; and if it be substituted for u in equation (1) of the preceding Article the result will be found to agree with equation (2).

Consequently W, the n^{th} Laplace's function, is also the $-(n+1)^{\text{th}}$ function.

Hence u admits of expansion in two independent series, one comprising only positive powers of r, and the other only negative powers; and the corresponding terms of these two series will have the same forms of W. We say *forms* because W will contain at least two independent terms, the equation (2) being of the second order.

129. In Art. 125 we have found the following subgeneral integral of the equation

$$0 = \cos^2\theta \,.\, \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right)u + \left(\cos\theta \,.\, \frac{d}{d\theta}\right)^2 u + \frac{d^2u}{d\phi^2},$$

$$u = \frac{A}{r}\left(\frac{\cos\theta}{r} \,.\, \epsilon^{i\phi}\right)^m,$$

in which m is a germ.

As a particular case take $m = n$; then the general term in the expansion of u will be

$$= \frac{A}{r^{n+1}} \,.\, (\cos\theta \,.\, \epsilon^{i\phi})^n.$$

But we have shewn that this term is equal to $Ar^{-n-1}W$; consequently one part of W is $(\cos\theta \,.\, \epsilon^{i\phi})^n$; and the other part will be found from this by writing $-(n+1)$ for n;

$$\therefore\ W = B(\cos\theta \,.\, \epsilon^{i\phi})^n + b(\cos\theta \,.\, \epsilon^{i\phi})^{-n-1} \ldots\ldots\ldots\ldots (1);$$

$$\therefore\ P = (Ar^n + ar^{-n-1})\,\{B(\cos\theta \,.\, \epsilon^{i\phi})^n + b(\cos\theta \,.\, \epsilon^{i\phi})^{-n-1}\},$$

and if we write m (a germ) for n, we may write u for P.

$$\therefore\ u = (Ar^m + ar^{-m-1})\,\{B\,(\cos\theta\,.\,\epsilon^{i\phi})^m + b\,(\cos\theta\,.\,\epsilon^{i\phi})^{-m-1}\}$$

$$= F(r\cos\theta\,.\,\epsilon^{i\phi}) + \frac{1}{r}f\,(r^{-1}\cos\theta\,.\,\epsilon^{i\phi})\ \ldots\ldots\ldots\ldots\ldots(2).$$

130. We may find W in another general form by means of Art. 126. For according to that Article we have

$$u = A\,(r\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,(r\epsilon^{i\theta})^m.$$

Assume $m = n + \frac{1}{2}$ as a particular case; the corresponding term in the expansion of u will be

$$r^n\,.\,(\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{i\,(n+\frac{1}{2})\theta}.$$

Hence the part of W corresponding to this is

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{i\,(n+\frac{1}{2})\theta}.$$

If in this we write $-(n+1)$ for n, we find the remaining part of W to be

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{-i(n+\frac{1}{2})\theta}.$$

Hence the complete value of W is

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\{B\epsilon^{i(n+\frac{1}{2})\theta} + b\epsilon^{-i(n+\frac{1}{2})\theta}\}$$

$$= (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\{B\cos\overline{n+\tfrac{1}{2}}\theta + b\sin\overline{n+\tfrac{1}{2}}\theta\}\ \ldots\ldots(1),$$

and the value of the general term in the expansion of u corresponding to this will be

$$= (Ar^n + ar^{-n-1})\,W\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Thus we have in this and the preceding Article found two distinct forms of Laplace's functions.

We may write m (a germ) for n, and the results will then take the form of arbitrary functions, and (2) will be equal to u.

Exceptional case of Laplace's Functions.

131. This occurs when the value of n is such as renders $n(n+1)=0$, for then the first term of equation (1) of Art. 127 vanishes, and the equation for the determination of W takes the following form,

$$0=\left(\cos\theta\,.\,\frac{d}{d\theta}\right)^2 W+\frac{d^2W}{d\phi^2}.$$

Assume τ such a function of θ as shall satisfy the equation $d\theta=\cos\theta\,.\,d\tau$;

$$\therefore\ \tau=\log\tan\left(\frac{\pi}{4}+\frac{j\theta}{2}\right),$$

$$\text{and } 0=\frac{d^2W}{d\tau^2}+\frac{d^2W}{d\phi^2};$$

$$\therefore\ W=F(\tau+i\phi)$$

$$=F(e^{\tau+i\phi})$$

$$=F\left(\epsilon^{i\phi}\tan\overline{\frac{\pi}{4}+\frac{j\theta}{2}}\right)\ \dots\dots\dots\dots\dots(1).$$

Also $Ar^n+ar^{-(n+1)}$ becomes $A+\frac{a}{r}$ when $n=0$;

$$\therefore\ u=\left(A+\frac{a}{r}\right)F\left(\epsilon^{i\phi}\tan\overline{\frac{\pi}{4}+\frac{j\theta}{2}}\right)\ \dots\dots\dots\dots(2).$$

This gives us that portion of u which corresponds to two terms of its expansion in integer powers of r, viz., A and $\frac{a}{r}$.

The algebraic signs of j and i are independent in these results.

132. To integrate the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=u.$$

We may assume the following as the form of the general exponential integral,

$$u = A\epsilon^{NMx+Nmy+nz},$$

subject to the two conditional equations

$$M^2 + m^2 = 1, \text{ and } N^2 + n^2 = 1;$$

$$\therefore\ u = A\epsilon^{nz} \,.\, \epsilon^{M(Nx)+m(Ny)}.$$

We may eliminate M and m from this integral, thereby obtaining the following general subintegral form,

$$P = \epsilon^{nz}(x+iy)^{-\frac{1}{2}}\epsilon^{jN\sqrt{(x+iy)(x-iy)}}$$

$$= (x+iy)^{-\frac{1}{2}}\epsilon^{nz+jN\sqrt{x^2+y^2}}.$$

By the same method we now eliminate n and N, and obtain the following as the first subintegral,

$$P = (x+iy)^{-\frac{1}{2}} \,.\, (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}}\epsilon^{j\sqrt{x^2+y^2+z^2}}$$

$$= (x+iy)^{-\frac{1}{2}}(\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}}\epsilon^{jr} \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$= (r\cos\theta \,.\, \epsilon^{i\phi})^{-\frac{1}{2}}(r\epsilon^{\pm i\theta})^{-\frac{1}{2}}\epsilon^{jr}$$

$$= r^{-1}\epsilon^{jr}(\cos\theta \,.\, \epsilon^{i\cdot\overline{\phi\pm\theta}})^{-\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Written out more fully this is equivalent to the following,

$$u = \frac{1}{r}(A\epsilon^{r} + a\epsilon^{-r})\sqrt{\sec\theta}\left(B\cos\frac{\phi\pm\theta}{2} + b\sin\frac{\phi\pm\theta}{2}\right) \ldots\ldots(3).$$

It is not to be forgotten that the form of the proposed equation shews that in integral (1) y and z are interchangeable.

In (3) there are no interchangeable variables.

133. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = u.$

We assume the following as a form which is equivalent to the general exponential integral,

$$u = A\epsilon^{\mu(Lx+My+Nz)}\cos\lambda\,(lx+my+nz+B) \ldots\ldots\ldots(1),$$

subject to the following condition (which the two disposable constants μ, λ enable us to assume)

$$(L+il)^2 + (M+im)^2 + (N+in)^2 = 0.$$

This resolves itself into the two following independent equations,

$$\left.\begin{array}{rl} & L^2+M^2+N^2=l^2+m^2+n^2 \\ \text{and} & Ll+Mm+Nn=0 \end{array}\right\}\ldots\ldots\ldots\ldots\ldots(2).$$

If the above value of u be substituted in the proposed equation we find that the following relation must hold good between μ and λ,

$$\mu^2-\lambda^2=1.$$

Hence if c be an extemporized germ this condition will be satisfied by assuming

$$\mu=\tfrac{1}{2}(c+c^{-1}) \text{ and } \lambda=\tfrac{1}{2}(c-c^{-1})\ldots\ldots\ldots\ldots(3).$$

Hence the integral form in (1) is fully determined.

Again, let $S=Lx+My+Nz$, and $T=lx+my+nz$, subject to the equations (2);

$$\begin{aligned} \therefore\ u &= A\epsilon^{\mu S}\cos\lambda T \\ &= A\epsilon^{\mu S+i\lambda T} \\ &= A\epsilon^{\frac{1}{2}(c+c^{-1})S+\frac{i}{2}(c-c^{-1})T} \\ &= A\epsilon^{c.\frac{1}{2}(S+iT)+c^{-1}.\frac{1}{2}(S-iT)}. \end{aligned}$$

Eliminating c from this integral we find the following form of the general subintegral,

$$P=(S+iT)^{-\frac{1}{2}}\,\epsilon^{i\sqrt{S^2+T^2}}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

134. The integrals of the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+u=0$$

may be deduced from the above integrals by changing the algebraic sign of c^{-1} but not that of c.

135. To integrate $\dfrac{d^2u}{dx^2}+\dfrac{d^2u}{dy^2}+\dfrac{d^2u}{dz^2}=\dfrac{du}{dt}$.

The general exponential integral may be put in the following form,

$$u=A\epsilon^{Lx+My+Nz+(L^2+M^2+N^2)t},$$

L, M, N being independent germs;

$$\therefore\ u=A\epsilon^{L^2t+Lx}\,.\,\epsilon^{M^2t+My}\,.\,\epsilon^{N^2t+Nz}.$$

The germs being all eliminated by Art. 64, we obtain the following form of the first subintegral,

$$P = t^{-\frac{1}{2}} \epsilon^{-\frac{x^2}{4t}} . t^{-\frac{1}{2}} \epsilon^{-\frac{y^2}{4t}} . t^{-\frac{1}{2}} \epsilon^{-\frac{z^2}{4t}}$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}}.$$

We may pass from this to the general integral in the following manner:

$$u = t^{-\frac{1}{2}} \epsilon^{-\frac{z^2}{4t}} F\left(\frac{d}{dx} + i\frac{d}{dy}\right) \epsilon^{-\frac{x^2+y^2}{4t}} + t^{-\frac{1}{2}} \epsilon^{-\frac{y^2}{4t}} f\left(\frac{d}{dx} + i\frac{d}{dz}\right) \epsilon^{-\frac{x^2+z^2}{4t}}$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} F\left(\frac{x-iy}{t}\right) + t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} f\left(\frac{x-iz}{t}\right)$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} \left\{ F\left(\frac{x+iy}{t}\right) + f\left(\frac{x+iz}{t}\right)\right\} \text{.......................(1).}$$

136. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = \dfrac{d^2u}{dt^2}$.

The general exponential integral is

$$u = A\epsilon^{L(MNx+mNy+nz+jt)},$$

the germs being subject to the following conditions:

$$M^2 + m^2 = 1, \text{ and } N^2 + n^2 = 1;$$

$$\therefore\ u = A\epsilon^{L(nz+jt)} . \epsilon^{M(LNx)+m(LNy)}.$$

Eliminating M and m we find the following subintegral,

$$P = \epsilon^{L(nz+jt)} . (x+iy)^{-\frac{1}{2}} \epsilon^{LN\sqrt{x^2+y^2}}$$

$$= \epsilon^{Ljt} (x+iy)^{-\frac{1}{2}} \epsilon^{n(Lz)+N(L\sqrt{x^2+y^2})};$$

and eliminating N and n in the same manner we find the following subintegral,

$$P = \epsilon^{Ljt} (x+iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}} \epsilon^{L\sqrt{x^2+y^2+z^2}}$$

$$= (x+iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}} \epsilon^{L(jt+r)};$$

$$\therefore\ u = (x+iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}} F(r+jt) \ldots\ldots\ldots\ldots (1)$$

$$= r^{-1} (\cos\theta \,.\, \epsilon^{i(\phi\pm\theta)})^{-\frac{1}{2}} F(r+jt)$$

$$(2) \ldots\ldots = \frac{\sqrt{\sec\theta}}{r} \left(A \cos\frac{\phi\pm\theta}{2} + B \sin\frac{\phi\pm\theta}{2}\right) \{F(r+t) + f(r-t)\}.$$

137. Laplace's equation is rendered important beyond most other equations by the circumstance that many problems of great interest in various branches of Natural Philosophy lead to it, and for their perfect solution render a knowledge of its general and complete integral a matter of necessity; for without that knowledge the investigator is obliged to assume some particular integral known to him, and this he fixes upon as being likely to answer the object he has in view. But this is a method which cannot but limit the generality of his results, and so far limit their authority in any case of appeal.

We shall, therefore, conclude this Essay with the following summary of the method and principles by which we have been enabled to accomplish the complete integration of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

(1). The form of this equation allows all the independent variables to take independent minor germs.

(2). From this we learn that any general integral of it may be perfectly expressed in the form of an infinite series in which the powers of x, y, z are positive integers.

(3). From this it follows that, subject to the condition

$$0 = L^2 + M^2 + N^2,$$

the quantities L, M, N being otherwise arbitrary, and not functions of any of the independent variables, the following is a complete and perfect integral of the above equation of Laplace,

$$u = A\epsilon^{Lx+My+Nz}.$$

When we call this the general integral of Laplace's equation, we must remember that the word *general* as thus used is dependent for its propriety on the fact that L, M, N are indeterminate quantities and not mere arbitrary constants.

The equation $y^2 = x$ may on the same principle be called a "parabola"; but when it is so called we assume that x and y are indefinite quantities that simultaneously belong to every individual point of the parabolic curve: for as x, y represent simultaneously all the values they can possibly have that are consistent with the equation $y^2 = x$, they simultaneously represent the co-ordinates of every point of the parabola.

On precisely the same principle we say that L, M, N simultaneously represent all the values that can be given to them which are consistent with the equation $L^2 + M^2 + N^2 = 0$; and accepting their significance in this general sense we denominate the integral above written the *general exponential integral* of Laplace's equation, just as we call $y^2 = x$ a *parabola.*

(4). There are many systems or sets of values of L, M, N in terms of two independent germs which will satisfy the equation $L^2 + M^2 + N^2 = 0$. Each set of such values will give a general integral in a form answering to that particular set by means of which it is obtained. Examples of this occur in the preceding Articles.

The following set of values of L, M, N containing two germs only, leads to a simple result;

$$\begin{aligned}&2Lx + 2My + 2Nz \\ &\qquad = (L - iM)(x + iy) + (L + iM)(x - iy) + 2Nz.\end{aligned}$$

Let $H^2 = L - iM$, and $K^2 = L + iM$;

$$\therefore\ H^2K^2 = L^2 + M^2 = -N^2;$$

$$\therefore\ N = iHK.$$

And $2Lx + 2My + 2Nz = H^2(x + iy) + K^2(x - iy) + 2iHKz$;

$$\therefore\ u = A\epsilon^{(H^2+K^2)x} \cos\{(H^2 - K^2)y + 2HKz + B\} \ldots (a).$$

$$\text{Also } u = A\epsilon^{H^2(x+iy)+2HKiz+K^2(x-iy)}.$$

From this form of u we may first eliminate H and then K, which will give one part of u's first subgeneral integral. The other part of that same subgeneral will be obtained by first eliminating K and then H. The result of elimination is the complete first subgeneral

$$= (x+iy)^{-\frac{1}{2}} F\left(\frac{x^2+y^2+z^2}{x+iy}\right).$$

The second subgeneral may be obtained from this by differentiating with $\frac{d}{dz}$. (See Art. 125.)

(5). We are tempted to pronounce the integral just found to be perfectly general, and so it is in one sense, because it is the type of the missing terms. But in another sense it is not the complete general integral, for the form of the differential equation of which it is the integral shews that in the complete general integral x, y, z must be interchangeable; they must therefore be made to enter the general integral in a symmetrical form.

This we may accomplish by means of Art. 121, and the result will be that we shall have the following instead of the integral in (4), (see Art. 125, equat. 1).

$$ru = F\left(\frac{Ax+By+Cz}{r^2}\right),$$

A, B, C being germs subject to the equation

$$A^2+B^2+C^2=0.$$

We may therefore write L, M, N instead of A, B, C.

But in (3) we have the following form of integral in terms of the same quantities,

$$\begin{aligned} u &= A\epsilon^{Lx+My+Nz} \\ &= F(Lx+My+Nz). \end{aligned}$$

Hence the complete form of the general integral of Laplace's equation which involves x, y, z in a manner which allows

x, y, z to be interchanged without affecting its form is the following:

$$u = F(Lx + My + Nz) + \frac{1}{r} f\left(\frac{Lx + My + Nz}{r^2}\right) \text{ ...... } (\beta).$$

We may therefore announce this as the complete general integral of Laplace's equation in terms of x, y, z.

(6). We now make a change of the independent variables from x, y, z to r, θ, ϕ.

By this change the differential equation itself takes the following form:

$$\cos^2\theta \, . \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right) u + \left(\cos\theta \, . \frac{d}{d\theta}\right)^2 u + \frac{d^2u}{d\phi^2} = 0.$$

Also, since $2\cos\phi = \epsilon^{i\phi} + \epsilon^{-i\phi}$, and $2\sin\phi = i(\epsilon^{-i\phi} - \epsilon^{i\phi})$;

$$\begin{aligned} \therefore\ & 2Lx + 2My + 2Nz \\ &= 2r(L\cos\theta\cos\phi + M\cos\theta\sin\phi + N\sin\theta) \\ &= r\{(L + iM)\epsilon^{-i\phi} + (L - iM)\epsilon^{i\phi}\}\cos\theta + 2rN\sin\theta \\ &= r(K^2\epsilon^{-i\phi} + H^2\epsilon^{i\phi})\cos\theta + 2rN\sin\theta \\ &= \text{.........................} + 2riHK\sin\theta. \end{aligned}$$

If we now eliminate $\cos\theta$ and $\sin\theta$ by means of their exponential equivalents, we find

$$\begin{aligned} & 4(L\cos\theta\cos\phi + M\cos\theta\sin\phi + N\sin\theta) \\ &= \frac{4(Lx + My + Nz)}{r} \\ &= (H\epsilon^{\frac{i\phi}{2}} + K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\pm i\theta} + (H\epsilon^{\frac{i\phi}{2}} - K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\mp i\theta}. \end{aligned}$$

As H and K are independent germs, we may omit the factor 4 on the left hand.

Hence assuming Q to represent the right-hand member of this equation the integral in (β) may be thus expressed in terms of r, θ, ϕ, *and* two germs H, K,

$$u = F(rQ) + \frac{1}{r} f\left(\frac{Q}{r}\right) \text{ .... } (\gamma).$$

By comparing (4) and (6) with each other it will be seen that the set of germs arbitrarily adopted in (4) is forced on us in (6) by the assumption, not altogether peculiar to this Essay, of the following change of variables,

$$x = r\cos\theta\cos\phi,\quad y = r\cos\theta\sin\phi,\quad \text{and}\quad z = r\sin\theta.$$

(7). If by Art. 24 we express the arbitrary functions in (γ) by means of an extemporized germ m we find the following equivalent integral,

$$u = Ar^m Q^m + ar^{-m-1}Q^m.$$

It is shewn in Art. 129 that the two subgeneral integrals of the equation

$$0 = m(m+1)\cos^2\theta\,.\,Q^m + \left(\cos\theta\,.\,\frac{d}{d\theta}\right)^2 Q^m + \frac{d^2Q^m}{d\phi^2}$$

(which is the equation of Laplace's functions) are Q^m and Q^{-m-1};

$$\begin{aligned}\therefore\ u &= (Ar^m + ar^{-m-1})(BQ^m + bQ^{-m-1})\\ &= \{A(rQ)^m + B(rQ)^{-m-1}\} + \left\{\frac{C}{r}\left(\frac{r}{Q}\right)^{m+1} + \frac{D}{r}\left(\frac{Q}{r}\right)^m\right\}\\ &= F(rQ) + \frac{1}{r}f\left(\frac{Q}{r}\right),\ \text{by Art. 24.}\end{aligned}$$

Hence $(Ar^m + ar^{-m-1})(BQ^m + bQ^{-m-1})$ is exactly equivalent to the integral in (6); and consequently the general expression for Laplace's m^{th} function is the following,

$$m^{\text{th}}\ \text{function} = BQ^m + bQ^{-m-1}\ldots\ldots\ldots\ldots\ldots(\delta),$$

B and b being independent arbitrary constants.

The independence of B and b indicates that there are two distinct Laplace's m^{th} functions, viz. Q^m and Q^{-m-1}; the product of which is constant, i.e. independent of m, being equal to Q^{-1}; where $Q = (H^2\epsilon^{i\phi} + K^2\epsilon^{-i\phi})\cos\theta + 2iHK\sin\theta$

$$= (H\epsilon^{\frac{i\phi}{2}} + K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\pm i\phi} + (H\epsilon^{\frac{i\phi}{2}} - K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\mp i\theta}.$$

(8). There is an exception to (δ) corresponding to $m = 0$, or -1. This is pointed out in equat. 1 of Art. 131.

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THE ANCIENT AND MODERN

GEOMETRY OF CONICS

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AN INTRODUCTION

TO THE

ANCIENT AND MODERN

GEOMETRY OF CONICS

BEING A GEOMETRICAL TREATISE ON THE CONIC SECTIONS WITH
A COLLECTION OF PROBLEMS AND HISTORICAL NOTES
AND PROLEGOMENA

By CHARLES TAYLOR M.A.
FELLOW OF ST JOHN'S COLLEGE CAMBRIDGE

ἐν τῇ γεωμετρίᾳ πᾶσίν ἐστιν ὁδὸς μία

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PREFACE.

Geometry, at once ancient and modern, is the science of Euclid, Archimedes and Apollonius, of Kepler, Desargues, Newton and Poncelet. Geometrical processes have indeed been simplified and their applications greatly extended in recent times, but the modern methods may be traced to the ancient as their germ and source, and thus it remains in a sense still true that there is but one road for all, *ἐν τῇ γεωμετρίᾳ πᾶσιν ἐστιν ὁδὸς μία.* The modern infinitesimal calculus is an adaptation of the ancient method of exhaustions, the method of Descartes differs only in the manner of its application from that of Apollonius, the idea of perspective was already formulated by Serenus, and the principle of anharmonic section with the leading properties of transversals are found in the lemmas of Pappus to the lost three books of Porisms of Euclid. There is not however in the works of the Greek geometers any distinct foreshadowing of Kepler's doctrine of the infinite, of his principles of analogy and continuity, or of the theory of ideal chords and points, at length completed by Poncelet's discovery of the so-called circular points at infinity in any plane.

In the present as in a former work (1863) I have commenced with an elementary treatment of the general conic *in plano*, following out a suggestion made by Professor Adams in a course of lectures on the Lunar Theory delivered in 1861. This department of the subject has now been made more complete with the help of the Eccentric Circle, the characteristic feature of a masterly though neglected work of Boscovich. In the chapter on the Cone the focal spheres are more fully discussed, and the angle-properties of the sections as well as their metric properties are deduced. The chapter on Orthogonal Projection contains proofs of Lambert's theorem* in elliptic motion. To the chapter on Conical Projection is appended some account of the homographic method of Reversion, which springs out of the above mentioned construction of Boscovich.

---

* It has been remarked that, so far as relates to the parabola, Lambert's theorem is implicitly contained in Newton's *Principia* lib. III. lemma 10. See Lagrange *Mécanique Analytique* tome II. p. 28, ed. 3 (1853—5); Brougham and Routh *An analytical view of Sir Isaac Newton's Principia* p. 436 (Lond. 1855).

Abundant references will be found to the works of authors to whom I am indebted. Suffice it here to add that my warmest thanks are due to the Reverend Professor Richard Townsend, F.R.S., Fellow of Trinity College, Dublin, who has been at all times ready, in the midst of pressing engagements, to aid me with his criticism and advice, and has from first to last shewn as great an interest in the work now brought to a close as if it had been his own.

C. TAYLOR.

St. John's College,
December 31, 1880.

ERRATA.

Page 42, Ex. 73	for	is	read	varies as.
Page 82, line 25	„	1657	„	1710.
Page 136, note*	„	Le Sueur	„	Le Seur.
Page 194, Scholium	„	Erastosthenes	„	Eratosthenes.
Page 206, Ex. 545	„	two or more	„	two.
Page 257, line 29	„	*Dynamics*	„	*Dynamic.*

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PROLEGOMENA.

SECTION I.

GEOMETRY BEFORE EUCLID.

N.B. *References within square brackets [ ] occurring in the Prolegomena are to the pages of the present work.*

§1. The science of Geometry, as its name suggests, was developed from the art of land surveying, to which ancient testimony likewise refers us for its origin. The practice of systematic land measurement is said to have been forced upon the Egyptians by the annual rise and fall of the river Nile, which from time to time left portions of land that had been high and dry submerged, or *vice versa*, so that the owners were unable to distinguish what belonged to each. Thus writes Hero[*] the elder, of Alexandria, and to the same effect Herodotus (II. 109), Diodorus Siculus and Strabo, as cited by Bretschneider in his excellent monograph on the history of geometry before Euclid.[†] Whether the Nile altogether played the part attributed to it in the advancement of science is matter of question, but it may be conceded to the concurrent testimony of ancient writers that the Egyptians had laid the foundation of concrete fact upon which the superstructure of Greek abstract geometry was to be reared.

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[*] Heronis Alexandrini *Geometricorum et Stereometricorum reliquiæ*, p. 138 ed. Hultsch (Berlin 1864). He flourished within the period B.C. 285–222, or later.

[†] *Die Geometrie und die Geometer vor Euklides* (Leipzig 1870). See also Dr. Allman's paper on Greek Geometry from Thales to Euclid, in *Hermathena* vol III. 160–207 (Dublin 1877).

Eudemus B.C. 340–320.

§ 2. Eudemus of Rhodes (a disciple of Aristotle and an immediate predecessor of Euclid) was the primary authority on the early history of mathematics;* but his writings on the history of geometry and astronomy, which appear to have been composed in a philosophical spirit, now no longer survive, except so far as they are embodied in the still extant works of his successors. The important list of early geometers given by Proclus Diadochus† (A.D. 412–485) in his commentary on the first book of Euclid's elements is not unreasonably thought to have been derived from Eudemus. The following is the substance of the passage, of which the original Greek with a German rendering may be found in the above mentioned work of Bretschneider, pp. 27–31. It is taken from lib. II. cap 4 of the commentary, which was written in four books:—

"Geometry is said by many to have taken its rise from the measurements rendered necessary by the obliteration of landmarks by the Nile. And it is nothing strange that this and other sciences should have arisen from practical needs, since there is a general tendency in things from imperfection to perfection, in accordance with which law we pass naturally from perception to reflection and thence to intellectual insight. As then the Phœnicians were led on from trade and barter to systematic arithmetic, so the Egyptians discovered geometry in the manner aforesaid.

First Thales went to Egypt and brought over this science to Greece. He made many discoveries himself and suggested the beginnings of many to his successors, apprehending some things more in the abstract but others in a limited and perceptional way. Next Ameristus, brother of the poet Stesichorus, became famed in geometry, as Hippias of Elis relates. Pythagoras, who succeeded them, transformed it into a liberal science, investigating its first principles and regarding theorems from the immaterial and intellectual standpoint. He it was who discovered the theory of irrational quantities and the construction of the regular solids. After him came Anaxagoras of Clazomenæ and Oenopides of Chios, who are mentioned by Plato as famed in mathematics. After them Hippocrates of Chios, who quadrated the lunule and was the earliest writer on the Elements, and Theodorus of Cyrene became eminent in geometry. Plato, who succeeded Hippocrates, greatly encouraged the study of mathematics and geometry by the frequent use of mathematical considerations in his philosophical writings. To this age also belong Leodamas of Thasos, Archytas of Tarentum and Theætetus of Athens. Younger than Leodamas were Neocleides and his disciple Leo, who added much to the work of their predecessors. Leo also composed a work on the Elements characterised by the greater number and importance of its propositions, and he assigned the limits within which a construction was possible. Eudoxus of Cnidus, an associate of the school of Plato and somewhat junior to Leo, increased the number of general theorems, added three new proportions to the three already known, and developed Plato's doctrine of the section (of a line), making use in his investigations of the method of geometrical analysis. **Analysis.** Amyclas of Heraclea, Menæchmus (a pupil of Eudoxus and contemporary with Plato) and his brother Dinostratus made geometry as a whole still more complete. Theudius of Magnesia, a writer on the Elements, and Athenæus of Cyzicus were greatly distinguished especially in geometry. These lived and worked together in the Academy. Hermotimus of Colophon carried on the discoveries of Eudoxus and Theætetus, and also wrote some things upon loci. **Loci.** Philip of Mende was led by Plato to study mathematics in relation to the Platonic philosophy. Thus far do the writers on the history of geometry bring the science.*

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* His contemporary Theophrastus also wrote something about mathematics, amongst a multitude of other subjects, according to the statement of Diogenes Laertius (lib. V. cap. 2).

† Notice the editions mentioned on p. [82], and the Latin edition of Barocius (Patavii 1560).

Not much junior to the above was Euclid, who compiled the Elements, putting in order many discoveries of Eudoxus, completing many of Theætetus, and replacing the former lax demonstrations by incontrovertible proofs. He lived in the reign of the first Ptolemy, in answer to whom he is reported to have said that there was no royal road to geometry.* He was therefore younger than the disciples of Plato, but elder than the contemporaries Eratosthenes and Archimedes. Being a Platonist, he made the construction of the Platonic bodies (or regular solids) the goal of his work upon the Elements."

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* If the history of Eudemus breaks off before Aristaeus, whose writings preceded Euclid's, we may conjecture that it was completed before 320 B.C. It is impossible to determine the precise dates of the early geometers.

Thales B.C. 640—548.

§3. Of the above mentioned early geometers Thales, Pythagoras, Hippocrates, Menæchmus and some others deserve to be noticed more particularly. Thales of Miletus, of Phœnician ancestry and the founder of the Ionian school of philosophy, was the first to naturalise the study of geometry amongst the Greeks. Visiting Egypt† as a trader, he brought back thence late in life to his native place such knowledge of geometry and astronomy as he had been able to pick up from the priests. He was born about the commencement of the 35th Olympiad, and died (according to one account) at the great age of 90 years, or upwards. His reputation was made once for all by the prediction—to what degree of accuracy we know not—of an eclipse of the sun, which duly came to pass (28th May, 585 B.C.)‡; and this well attested fact corroborates the statement of Diogenes Laertius (lib. I. cap. 1) that he first came to be styled σοφός in the archonship of Damasias. Although he is said by Proclus in general terms to have made many discoveries in geometry, the following alone are expressly attributed to him.‖ (1) The circle is bisected by its diameter. (2) The angles at the base of an isosceles triangle are equal to one another. (3) When two straight lines cut one another the vertical angles are equal. (4) A method of determining the distance of a ship at sea from the land, implying the knowledge of a theorem equivalent to Euclid I. 26. (5) The angle in a semicircle is a right angle.* And (6) a method for determining the heights of the pyramids from the lengths of their shadows, viz. at the moment when the sun is at an elevation of 45° above the horizon.†

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* The saying referred to (Bretschneider p. 163) is also attributed to Menæchmus, who is said to have replied to Alexander: "In the country, O king, there are roads ἰδιωτικαὶ καὶ βασιλικαί, but in geometry there is one road for all."

† The foreign travels of the early Greek philosophers are however sometimes thought to be attested by insufficient evidence. Cf. Renouf's *Hibbert Lectures* Lect. VI. p. 246 (London 1880).

‡ The Egyptians had doubtless supplied him with the facts on which his calculation was based. Diogenes Laertius states that they had observed more than 1200 eclipses of the sun or moon. See Bretschneider pp. 39, 52.

‖ See Proclus on Euclid I. def. 17 and props. 5, 15, 26 (Thos. Taylor's Proclus vol. I. 165; II. 54, 96, 143); Diogenes Laertius lib. I. cap. I. §§ 3, 6.

The fact that a theorem was attributed to Thales by his successors does not altogether exclude the supposition that he had himself received it from the Egyptians; and accordingly it has been thought that the second only of the above theorems was in reality discovered by him. The theorem (5) may have been arrived at by the Egyptian geometers by supposing first a square and then any rectangle inscribed in a circle to be turned about within it; and it is impossible to lay much stress on (1) or (3). The method (4) if actually known to Thales was probably discovered by him, but if (as has been conjectured) he was acquainted only with the case of the right-angled triangle,‡ his knowledge of this, as also of (6), may very well have been derived from the Egyptians. On the whole we may conclude that he probably made some advance towards that abstraction by which the Greek geometry, in contrast with the Egyptian, was to be characterised; but more than this cannot safely be affirmed until we are better informed as to the "many things" which he is said to have discovered for himself. Thales was acquainted with the globular form of the earth, which was held by his school to be at the centre of the world.

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* This is of course the meaning of the statement that he was the first *to inscribe a right angled triangle in a circle.*

† Plutarch in his *Symposium* states the method in a form requiring a knowledge of similar triangles and applicable at any time of the day. The most trustworthy part of the story is that the method in its simpler form was used *in Egypt*. It would serve for an obelisk, but scarcely for a pyramid.

‡ This view is taken by Bretschneider (p. 43), who attributes (2) only to Thales himself. For a more appreciative estimate of his contributions to geometry see *Hermathena* vol. III. 173.

Pythagoras B.C. 580–500.

§4. The name of Pythagoras of Samos next arrests our attention. Although the date of his birth and his age at his death are variously given, he was doubtless a young man when Thales, of whom he was regarded as the successor, died. At the instigation of Thales he visited Egypt, where he resided many years, learned the Egyptian language and received instruction from the priests. After this he is said to have visited or been carried captive to Babylon. Returning to Samos at a time when Ionia had lost her independence, he migrated thence to Crotona in Magna Græcia, where he gathered round him his exclusive brotherhood and became the founder of the famed Italian school; but in course of time he was banished by the democratic party, and died shortly afterwards at Metapontum. We proceed to notice some of the chief discoveries in mathematics attributed to him, remarking however that it is impossible to distinguish with certainty between the discoveries of the master and his scholars, since the doctrines of the sect were in the first instance communicated only to its members, and when they came at length to be divulged it was the practice to attribute everything to Pythagoras himself. Hippasus, who offended against this rule, was lost at sea for his impiety (Iamblichus *Vit. Pythag.* cap. 18). He had taken credit to himself for the construction of the sphere circumscribed to a regular dodecahedron (τὴν ἐκ τῶν δώδεκα πενταγώνων), whereas everything belonged to Him (εἶναι δὲ πάντα Ἐκείνου), "for so they call Pythagoras, and not by his name."

a. The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the sides containing the right angle.

In honour of this great discovery, as also on some other occasions, Pythagoras is related to have offered a sacrifice. There is no evidence to support the conjecture that the theorem was known in its generality to the Egyptians, although it must be allowed to partake of an Egyptian character,* and may have been first proved by an application of the Egyptian method. The Egyptians were acquainted with the fact that the triangle whose sides contain 3, 4, and 5 units of length respectively is right angled (Plutarch *De Iside et Osiride* cap. 56), which is a special case of the theorem of Pythagoras; and they must also have been familiar with the still simpler case in which the right angled triangle is isosceles. To prove it for this case, let one square be supposed to be inscribed symmetrically in another. Then it is easily seen that the four triangles at the corners may be fitted together so as to form two squares, the sum of which is equal to the area of the inscribed square; whence the theorem at once follows for the case in question.

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* The Egyptian geometry had very little that was of an abstract or general character, but consisted mainly in the computation of areas or volumes, and in such special constructions as are required for geometrical drawing. Cf. Eisenlohr's edition of the Rhind papyrus, published under the title *Ein mathematisches Handbuch der*

By some such method of dissection of figures the general theorem also was perhaps arrived at; and that it was not in the first instance proved by the method of Euclid might have been taken for granted, even without the express statement of Proclus in his comments upon Euclid I. 47. To prove the theorem generally,* let one square inscribed in another divide each side of the latter into segments equal to a and b respectively, and let the side of the inscribed square be equal to h. Then the whole figure, being made up of h^2 and the four triangles, is evidently equal to $h^2 + 2ab$. Next, by considering the figure of Euclid II. 4 (omitting the diagonal), we see that the outer square may also be cut up into two rectangles, each equal to ab, and two squares equal to a^2 and b^2 respectively. Hence it follows that

$$h^2 + 2ab = a^2 + b^2 + 2ab,$$

and therefore h^2 is equal to $a^2 + b^2$, or the square on the hypotenuse of one of the triangles is equal to the sum of the squares upon its sides. Thus the theorem is shewn to be true for any right angled triangle. Pythagoras added a rule for finding triads of integers a, b, h satisfying the relation $a^2 + b^2 = h^2$. The problem of the three squares would naturally suggest an analogous problem relating to cubes; and to a special case of the

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alten Aegypter (Leipzig 1877), and accompanied with a German translation in a separate volume.

* The proof here given is taken from Bretschneider's *Die Geometrie &c.* pp. 81—2.

latter, the Duplication of the Cube, we shall see that the further progress of mathematics was indirectly to a very great extent due.

b. The three angles of a triangle are together equal to two right angles.

Eudemus, according to a statement of Proclus on Euclid I. 32, ascribes the discovery of this theorem to the Pythagoreans, together with a general proof of it not unlike that given by Euclid. But since Eutocius,* on the authority of Geminus, asserts that the ancients were accustomed to prove it separately for the equilateral, isosceles and scalene triangles, whilst only the later geometers proved it generally once for all, it has been conjectured that its truth may have been known even to the Egyptians, the general proof only being Pythagorean. Although the fact that the area about a point can be filled up by equilateral triangles, squares or regular hexagons, and by no other regular figures, is said by Proclus (on Euclid I. 15, Cor.) to be a Pythagorean discovery, the positive part of it must have been observed by the Egyptians, who must therefore have known that the three angles of an equilateral triangle are together equal to two right angles. They may also have inferred the same for any right angled triangle regarded as the half of a rectangle; and it would then remain only to observe that an isosceles or scalene triangle may be divided into two right angled triangles. By some such process the theorem (by whomsoever discovered) may have been first arrived at; or it may have been shewn experimentally that the six angles of any two triangles exactly fit into the area about a point.

c. The regular polyhedra.

We have seen that the construction of the regular solids was attributed to Pythagoras by Proclus, doubtless upon the authority of Eudemus. Of these five figures the tetrahedron, the cube and the octahedron were known to the Egyptians and occur in their architecture; but it does not appear that they were acquainted with the icosahedron or the dodecahedron. In the construction of the last mentioned solid the regular pentagon is required; and with this the Pythagoreans were familiar, since they used the starred pentagram,[*] formed by producing its alternate sides to meet, as a secret token of recognition symbolical of *ὑγίεια*, the letters *υ*, *γ*, *ι*, *θ* (= *ει*), *α* being written at the five angles of the figure. Moreover we have seen [p. xxii] that the knowledge of the dodecahedron was said to have been possessed and divulged by the Pythagorean Hippasus. The regular solids were also called the "cosmic figures," the dodecahedron being taken to represent the material world, and the remaining four its elements of earth, air, fire, and water.[†]

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* Halley's Apollonius p. 9; Bretschneider's *Die Geometrie &c.* p. 14.

d. The application of areas.

The *παραβολή* or application of areas is attributed in general terms to the Pythagoreans (Proclus on Euclid I. 44), and also to Pythagoras in particular, who is said to have sacrificed an ox, *ἐπὶ τῷ διαγράμματι*,[‡] where the reference is either to this discovery or to that of the theorem of the three squares (Euclid I. 47). An area, according to Proclus, was said to be *applied* to a right line when an equal area was described upon the line as base; but the term was also used more generally to include the cases in which the base of an area placed upon a given line was in excess (*ὑπερβολή*) or defect (*ἔλλειψις*) of the line to which it was "applied." Although it has not been made out wherein consisted the importance of the discovery in the hands of the Pythagoreans, we shall see that it played a great part in the system of Apollonius, and that he was led to designate the three conic sections by the Pythagorean terms Parabola, Hyperbola, Ellipse. It is not however to be thought that Pythagoras or his school had any acquaintance with these curves, although, through a misunderstanding and consequent misreading of the term *παραβολή* of areas, he has been supposed to have anticipated Archimedes (whose name, as it happens, follows in the immediate context) in his discovery of the quadrature of the parabola.

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[*] On the *Polygones étoilés* see Chasles *Aperçu historique* pp. 476—87 (1875).

[†] See Boeck's *Platonica corporis mundani fabrica &c.* (Heidelberg 1809).

[‡] Plutarch on Epicureanism, cap. 11. See Plut. *Op.* IV. 1838, ed. Dübner (Paris 1841), where the misreading *περὶ τοῦ χωρίου τῆς παραβολῆς* occurs.

e. Incommensurability and proportion.[*]

To Pythagoras, as we have seen [p. xix], was attributed the theory of incommensurable magnitudes, which may be regarded as a corollary from his theorem of the three squares (Euclid I. 47). He was also acquainted with the doctrine of proportion, and is related by Plutarch to have solved the problem, to describe a rectilinear figure equal to one and similar to another given figure (Euclid VI. 25), and on this occasion also to have offered a sacrifice; but whether he completed the theory of proportion by extending it to the case of incommensurable magnitudes we are unable to say. Iamblichus states that in the time of Pythagoras three kinds of proportion only were known, viz. "the arithmetic, the geometric, and in the third place the subcontrary, as it was then called, but which was afterwards called the harmonic by the associates of Archytas and Hippias." Further on he remarks of the so called "most perfect" or "musical" proportion,

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b,$$

which combines in itself the three former, that it was said to be a discovery of the Babylonians and to have been brought by Pythagoras to Greece. To him belongs the credit of combining the Eastern science of arithmetic, which he esteemed so highly, with the Egyptian science of geometry.

f. The circle.

Iamblichus,[†] giving however no details, says that although Aristotle may not have squared the circle the problem was at any rate solved by the Pythagoreans. This problem, as we learn from the Rhind papyrus (ed. Eisenlohr vol. I. 98, 117), had already engaged the attention of the Egyptians, who estimated the circle on a diameter of nine units to be equal to the square on a line containing eight, thus making π equal to $3\frac{13}{81}$. Having regard to the perfect symmetry of the sphere and the circle, Pythagoras speaks of the one as the most beautiful of solids and the other of plane figures (Diogenes Laertius lib. VIII. cap. 1); but there is no ground for the statement sometimes made that he speaks of the circle as the maximum plane figure having a given perimeter and of the sphere as the maximum solid having a given surface.*

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[*] See Bretschneider pp. 75, 83.

[†] See the extract from Simplicius on Aristotle given by Bretschneider, p. 108.

Loci.

§ 5. The further development of geometry was due in great measure to repeated attempts to square the circle, to trisect an angle and to duplicate the cube, which led to the discovery of various geometrical loci. Thus Pappus† ascribes to Dinostratus and Nicomedes the use of the quadratrix for squaring the circle; and Proclus (on Euclid I. 9) relates that Nicomedes trisected a given angle by means of the conchoid (of which he had himself discovered the genesis and investigated the properties), others used the quadratrix of Hippias or Nicomedes for the same purpose, whilst others by means of the spiral of Archimedes divided a given angle in any given ratio. The problem of the duplication of the cube, as we shall see, was solved by the intersections of parabolas or other conics, and perhaps actually led to the discovery of the sections of the cone. It is to be noticed that the construction of such a curve as, for example, the quadratrix implies the conception of the idea of a *Locus*, of which before the time of the above mentioned Hippias of Elis, a contemporary of Socrates, there is no trace, although the idea must have presented itself in a rudimentary form in the construction of a circle by the most obvious method. The earliest writer on loci was Hermotimus of Colophon, one of the successors of Eudoxus [p. xix].

Hippocrates B.C. 450–430.

§ 6. Hippocrates of Chios is referred to by Aristotle (*Ethica* Eudem. VII. 14) in illustration of the fact that there are persons who are wanting in intelligence in some respects although not in others. The geometer shewed his simplicity by allowing himself to be defrauded by the douaniers of Byzantium; or, according to Johannes Philoponus, he was robbed by pirates, went to Athens to obtain redress, there frequented the schools of the philosophers and made such progress in geometry that he ventured to attack the problem of the quadrature of the circle. It is related by Iamblichus that Hippocrates was expelled from the school of the Pythagoreans for having taught for hire.

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* For an actual mention of these theorems see Pappus *Collectio* lib. V. (vol. I. pp. 316, 350 ed. Hultsch).

† *Collectio* lib. IV. prop. 25 (vol. I. 251, ed. Hultsch).

[The Delian problem.

Hippocrates is celebrated as having reduced the problem of the duplication of the cube to the simpler form in which it was thenceforth attempted by geometers. By the duplication of the cube was signified the construction of a cube of twice the volume of a given cube: a problem which may possibly have first presented itself in architecture, or may have arisen speculatively in the course of an attempt to find an analogue in space to the Pythagorean property of squares (Euclid I. 47). Eutokius, commenting upon the second book of Archimedes *De Sphœra et Cylindro*, adduces a series of solutions of the problem, including the solution of Eratosthenes given in his letter to king Ptolemy II. together with a twofold tradition as to the origin of the problem (Archimedis *Op.* p. 144, ed. Torelli). Minos of Crete, according to one of the ancient tragedians, ordered a sepulchre for his son Glaucus, and then, deeming the proposed dimensions of the (cubical) structure inadequate, directed the architect to make it exactly twice as large. At a later period—so the story runs—the people of Delos, in time of pestilence, were commanded to construct a new cubical altar twice as large as one already existing, and accordingly at their request the philosophers of the Academy set to work to solve this "solid problem" [p. xxxiii], which was found to transcend the power of the known geometry of the straight line and circle. It involved in effect the extraction of a cube root, or the solution of the cubic equation $x^3=2a^3$. Hippocrates reduced it to the problem of finding a pair of mean proportionals to two given magnitudes a and b, that is to say, of determining x and y so as to satisfy the relations,

$$a:x=x:y=y:b,$$

which evidently imply also the relation $x^3 = a^2b$. It does not appear that he himself carried the solution any further, but the problem was afterwards attacked in this form by geometers, and in particular it was solved by Menæchmus in two ways with the help of the conic sections, of which he was the discoverer. The problem of the duplication of the cube went by the name of the *Delian Problem* owing to the above mentioned tradition connecting it with one of the altars at Delos.

Quadrature of lunules.

Hippocrates is also celebrated as having, in his attempts to square the circle, quadrated the lunule contained by the circumscribed semicircle of an isosceles right angled triangle and the semicircle described outwards on one of its shorter sides as diameter. By an extension of his method it may be shewn that the circumscribed semicircle of a scalene right angled triangle contains with the semicircles described in like manner on its two shorter sides two lunules which are together equal to the area of the triangle; but it does not clearly appear that the theorem in this more general form is rightly ascribed to him. In his further attempts to square the circle, he succeeded only in shewing that the problem could be solved if the lunule bounded by an arc equal to a sixth part of the circumference and the semicircle described outwards upon the chord of the arc as diameter could first be squared. All this is fully discussed in a passage of Simplicius,* a commentator on Aristotle, which is given at length by Bretschneider, pp. 100–121. Simplicius gives a long extract from Eudemus, interspersed with references of his own to Euclid, from which it appears that Hippocrates made use of the following propositions in his researches. (1) Circles are to one another as the squares of their diameters. (2) Similar segments (defined as those which are the same fractional part of the circumference) contain equal angles. And (3) similar segments are to one another as the squares of their bases. It is possible, as has been suggested, that by the angle in a segment he means the angle subtended by the chord of the segment at the middle point of its arc, not knowing that the angle subtended at any point of the arc is constant and equal to half the angle at the centre; although *prima facie* it would appear that, knowing already so much, he must have been acquainted with this also.

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* Simplicius, on Aristotle *De physica auscultatione*, fol. 12*a* (Venetiis 1526).

Antipho B.C. 440.

§7. The same passage of Simplicius contains an account of the method by which Antipho, perhaps the well known opponent of Socrates, attempted the quadrature of the circle. He first inscribed say a square in the circle, then (bisecting each quadrant) an octagon, then a 16-gon, and so on continually, till at length he supposed a regular inscribed polygon to be arrived at, having an infinity of infinitesimal sides, which was to be regarded as coincident with the circle. Although his principles were regarded as unsound by the ancient critics, he had in fact introduced the fundamental idea of infinitesimals into the geometry of curves, and had virtually proved (1) that the areas of circles are as the squares of their diameters—as his contemporary Hippocrates had also somehow arrived at; and (2) that their circumferences are proportional to their diameters. On the quadrature by Bryso, a contemporary sophist, who regarded the circle as intermediate to an inscribed and a circumscribed n-gon, and then applied the method of Antipho, see Bretschneider's *Die Geom. vor Eukl.* pp. 126 ff.

Plato B.C. 429–348.

§8. Plato, although not greatly distinguished for his own discoveries in geometry, became the founder of a school which was soon to carry the science to unknown heights. He indeed devised an organic solution of the problem of the two mean proportionals, depending upon a double application of a property of the right angled triangle, and gave a rule of his own for constructing right angled triangles having their sides commensurable (Proclus on Euclid I. 47); but he rendered far greater service to geometry by his systematic treatment of its definitions and primary ideas, and by the impulse which he gave to the study amongst his disciples by insisting upon a knowledge of it as a prerequisite for metaphysical speculation, writing up (as it is said) before his vestibule, *μηδεὶς ἀγεωμέτρητος εἰσίτω μοῦ τὴν στέγην.* To Plato are attributed the propositions Euclid VIII. 11, 12. One of his disciples Theætetus, who had been led by him to the study of incommensurable quantities in connexion with proportion, became the author of the propositions Euclid x. 9, 10. Another, Menæchmus, developed the germs of stereometry received from him and was led, in what way we can only conjecture, to the discovery of the conic sections, which is sometimes erroneously attributed to Plato himself, owing to a misunderstanding of the term τὴν τομήν, *the section*, in a passage quoted above from Proclus [p. xix], where it refers not to the cone but to the right line. Archytas of Tarentum, a contemporary of Plato, propounded a solution of the Delian problem, and is said to have been the first to apply the method of organic description to geometrical figures; a method which Plato (notwithstanding his own application of it as above mentioned) condemned, as tending to materialise geometry and bring it down from the region of eternal and incorporeal ideas. It was one of his sayings, τὸν θεὸν ἀεὶ γεωμετρεῖν, which Plutarch discusses in his *Quæst. Conviv.* lib. VIII. q. 2. Plato is said (Diogenes Laert. lib. III. cap. 1) to have introduced the method of geometrical analysis, and to Analysis. have communicated it to Leodamas of Thasos.

§ 9. Menæchmus,* a hearer of Eudoxus and contemporary Menæchmus with Plato, is expressly said by Proclus (on Euclid I. def. 4), B.C. 350—330. upon the authority of Geminus, to have been the discoverer of the conic sections, which were accordingly at first named after him the "Menæchmian triads" [p. 194]. He also applied them in two ways to the solution of the problem of the two mean proportionals [pp. 45, 189], to which the Delian problem had been reduced by Hippocrates of Chios. It remains to consider whether he in the first instance regarded the curves in question as plane loci or as sections of a cone. In favour of the former view it may be urged that, as geometers before and after him were led to the discovery of the quadratrix, the conchoid and other plane loci in their attempts to square the circle or trisect the angle, so Menæchmus may have discovered his triad of curves in considering by means of what loci* the construction of a pair of mean proportionals to two given magnitudes might be effected. This implies the use of the method of geometrical analysis, which was said to have been discovered by Plato [p. xxxi]; and accordingly we find that Eutokius, who gives in detail the two solutions by Menæchmus of the problem of the two means (Archimed. *Op.* pp. 141–2, ed. Torelli), represents him as having employed the method in both cases. But it is more important to notice that it was used by Eudoxus, of whom Menæchmus was a hearer [p. xix].

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* The anecdote which brings Menæchmus into relation with Alexander the Great [p. xx] is consistent with the supposition that he was a younger contemporary of Plato.

(1) The problem being to find the two magnitudes x and y which with two given magnitudes a and b constitute the continued proportion

$$a : x = x : y = y : b,$$

it was seen that the relations $x^2 = ay$ and $y^2 = bx$ were to be satisfied. Being then, as we have seen reason to conclude, already familiar with the idea of a locus, Menæchmus had virtually discovered the parabola regarded as the plane locus determined by the relation $x^2 = ay$, and it was evident that by the intersection of two such curves the required construction could be effected [p. 45].

(2) In his second solution of the problem he makes use of a parabola and a rectangular hyperbola [p. 189], the latter curve being regarded as possessing the property that the product of the distances of any point on it from the asymptotes is constant; whence it is inferred by Bretschneider (*Die Geom. vor Eukl.* p. 162) that the asymptotes of the hyperbola must have been discovered very soon *after* the curve itself became known. But when we consider that the assumed relations,

$$a : x = x : y = y : b,$$

are evidently equivalent to $xy = ab$ and $x^2 = ay$, it commends itself as a not less simple hypothesis that, having already formed the conception of the curve $x^2 = ay$, Menæchmus was further led by the conditions of the problem to attempt the construction of the curve satisfying the relation $xy = ab$. This at any rate seems to be the only property of the hyperbola with which he can be safely assumed to have been acquainted. The ellipse does not occur in either of his solutions. To construct his loci he may now have had recourse in the first instance to the organic methods reprehended by Plato,* not at first perceiving that they could be more simply constructed by cutting the right cone by planes. It is less natural to suppose† that *after* the discovery of their genesis from the cone Menæchmus, or his followers, should have thought it necessary to trace them by mechanical appliances, of such a nature as to be almost immediately rejected and forgotten. But even if he never so traced them, he may still have discovered them as plane loci. Their actual description‡ was felt to be a difficulty many centuries later.

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* We have seen, from his acquaintance with the quadratrix, that Dinostratus, the brother of Menæchmus—not to mention Hippias of Elis in the preceding century—must have been familiar with the idea of a locus [p. xxvii].

Aristæus B.C. 320.

§10. The conic sections, in whatsoever way first discovered, soon came to be regarded as "solid loci," and problems which required them for their solution were called "solid problems." The first writer on the subject was Aristæus the Elder, who distinguished the three conics as the sections of the acute-angled, right-angled and obtuse-angled right cones respectively by planes drawn at right angles to their sides [p. 195]. He is said by Pappus‖ to have written five books of Conic Elements, and five (in continuation?) upon Solid Loci,§ thus preparing the way for the work of Euclid on Conics. He also instituted a comparison of the regular polyhedra,¶ to which Euclid may have been indebted in the thirteenth and last book of his Elements. We assign to Aristæus the date B.C. 320, to indicate that he was intermediate to Euclid and Menæchmus.

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* Plutarch *Quæst. Conviv.* lib. VIII. q. 2; *Vita* Marcelli, cap. 14.

† Bretschneider *Die Geom. vor Eukl.* p. 143.

‡ Eutocius (on Apollonii *Conica* I. 20, 21) remarks that it was often necessary, διὰ τὴν ἀπορίαν τῶν ὀργάνων, to describe a conic by points, and that this might be done by means of the relations $y^2 = px$, &c.

‖ *Collectio* lib. VII. § 29 &c. (vol. II. 672—6, ed. Hultsch).

§ Viviani [p. 221], in a *Secunda divinatio &c.* (Florent. 1701), attempted to restore the *Loca Solida* of Aristæus.

¶ See prop. 2 of the so-called 14th book of Euclid's Elements.

SECTION II.

FROM EUCLID TO SERENUS.

Euclid B.C. 300. §1. The birthplace of the geometer Euclid, sometimes confounded with his namesake of Megara, is unknown. He lived under the first Ptolemy (B.C. 323–284), about two centuries after the death of Pythagoras; and we find him established at Alexandria, "etwa im Jahre 308, als den ersten Mathematiker seiner Zeit."* Of the various lost works attributed to him we may mention (1) his treatise on *Conics*, which formed the nucleus of the great work of Apollonius, and (2) the three books of *Porisms*, to which we shall again refer in speaking of Pappus. His Στοιχεῖα or *Elements* was written in thirteen books, to which a fourteenth and a fifteenth (by Hypsicles of Alexandria) are sometimes appended. The books 1–6 are too well known to need description. Books 7–9 are on the properties of numbers; book 10 on incommensurable magnitudes; and books 11–13 on stereometry. Book 10 commences with the proposition, that *If from the major of two given magnitudes more than its half be taken away, and from the remainder more than its half, and so on continually; a remainder will at length be arrived at which is less than the minor given magnitude.* Since the minor given magnitude may be assumed to be as small as we please, the proposition is seen to embody the idea of convergent series and the principle of the "method of exhaustions." The book ends with the proposition that the diagonal and the side of a square are incommensurable. The 12th book contains applications of the method of exhaustions to plane and solid figures, and it is shewn that the areas of circles are as the squares of their diameters (prop. 2), every cone is the third part of a cylinder having the same base and altitude* (prop. 10), and the volumes of spheres are as the cubes of their diameters (prop. 18). In the 13th book, which is a sequel to the 4th, it is shewn that there are but five regular polyhedra, such as can be inscribed or circumscribed to a sphere. The *editio princeps* of the *Στοιχεῖα* was published at Basel in 1533 [p. 82]: the Arabic version at Rome in 1594. The Oxford Græco-latin edition, by David Gregory, of the extant works of Euclid was issued in 1703: it contains the Elements, Data, Introductio Harmonica, Sectio Canonis, Phænomena, Optica, Catoptrica, De Divisionibus Liber, De Levi et Ponderoso fragmentum. Notice also Peyrard's *Les Œuvres d'Euclide, en grec, en latin et en français* (Paris 1814–18).

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* See Moritz Cantor's *Euclid und sein Jahrhundert* p. 2 (Zeitschrift. f. Math. u. Physik. Suppl I. 1867).

To what extent Euclid was himself a discoverer we are unable to say, but in his Elements he is to be regarded mainly as a compiler. His system as a whole must however have been more or less original in its conception; and the best testimony to its superior method and completeness is the subsequent neglect and disappearance of the cognate works of his predecessors. But his work, although the most ancient on the *Στοιχεῖα* still surviving, must not be supposed always to preserve the most ancient methods of proof. Thus the theorem of Thales (Euclid I. 5) cannot have been first proved in the manner of Euclid;† whilst Proclus expressly states that the theorem of Pythagoras was not originally proved as in Euclid I. 47. It was also perhaps first shewn more briefly than by Euclid that circles are as the squares and spheres as the cubes of their diameters.

Archimedes B.C. 287–212.

§ 2. Archimedes of Syracuse was born in the year 287 B.C.‡ According to Plutarch (*Vita* Marcelli cap. 14) he was related to king Hiero, whilst Cicero on the other hand speaks of him as "humilem homunculum" (*Tusc. Disp.* lib. v. cap. 23). He was a master not only of geometry, but also of theoretical and applied mechanics. By his scientific conduct of the defence of Syracuse against the Romans the siege was protracted for two years, till at length the city was carried by a surprise from the land side, and Archimedes fell by the hand of a soldier (212 B.C.). His grave was marked by the figure of a sphere inscribed in a cylinder, in commemoration of his most cherished discoveries, and by that sign it was recognised by Cicero in the course of his quæstorship in Sicily. His works, according to the Græco-latin edition of Torelli (Oxon. 1792), are as follows:

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* This theorem, as we shall see, was discovered by Eudoxus [p. xxxviii].

† The more direct way of deducing it from prop. 4 is mentioned by Proclus, in connexion with the name of Pappus.

‡ Notices of the life and works of Archimedes (and of Apollonius) are contained in Cantor's *Euclid u. s. Jahrhundert*. See also Heilberg's article on his knowledge of the *Kegelschnitte* in the *Zeitschr. f. Math. u. Physik* (April 1880).

(1) *De Planorum Æquilibriis.** Two books, with the tract *Quadratura Paraboles* placed between them (pp. 1–60).

(2) *De Sphæra et Cylindro.* Two books (pp. 61–201).

(3) *Circuli Dimensio* (pp. 203–216).

(4) *De Helicibus* (pp. 217–255).

(5) *De Conoidibus et Sphæroidibus* (pp. 257–318).

(6) *Arenarius* (pp. 319–332).

(7) *De iis quæ in Humido vehuntur.* Two books, in Latin only (pp. 333–354).

(8) *Lemmata*, translated from the Arabic (pp. 355–361).

(9) *Opera mechanica*, ut cujusque mentio ab antiquis scriptoribus facta est (pp. 363–370).

We learn also from one of the scattered notices of Archimedes in the *Collectio* of Pappus (lib. v. § 34 vol. I. p. 352, ed. Hultsch) that he discovered thirteen semi-regular polyhedra, bounded by regular but not similar polygons—one of them, for example, by 20 triangles and 12 pentagons, another by 30 squares, 20 hexagons and 12 decagons. But his greatest achievements in geometry were his approximate quadrature and rectification of the circle, his quadrature of the parabola, and his applications of the method of exhaustions to the quadrics of revolution.

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* To this treatise and to (2) and (3) are appended the commentaries of Eutocius of Ascalon (540 A.D.).

Parabola quadrated.

§ 3. In the introduction to his treatise Τετραγωνισμὸς Παραβολῆς, as later scribes have entitled it, Archimedes remarks that none before him, so far as he knows, has attempted to quadrate the segment cut off by a right line from the "section of a right-angled cone," for so he calls the parabola [p. 195]. His theorem, which was first arrived at by mechanical considerations and afterwards proved by pure geometry, is stated as follows:

The segment contained by any right line and the section of a right-angled cone is equal to four-thirds of the triangle which has the same base and altitude as the segment.

a. In the mechanical proof he shews first that a triangle CDE suspended from a lever of equal arms AB and BC, so as to have its side DE vertical and in a line with the fulcrum B, is balanced by an area equal to one-third of its own suspended from A (prop. 7); and that the parallel sided trapezium cut off from the triangle CDE by two vertical lines drawn at horizontal distances h and k from B is balanced by an area suspended at A intermediate to $\frac{h}{BC}$ and $\frac{k}{BC}$ of the trapezium (prop. 13). Lastly, supposing a parabolic segment on BD as base to be suspended with its vertex downwards, he arrives at the required quadrature by successive applications of the foregoing theorems after the manner of the method of exhaustions.

b. The following is a summary of his second and purely geometrical proof of the same theorem (props. 20–24). If P be the vertex [fig. p. 58] and QQ' the base of a segment of a parabola, the triangle QPQ' is greater than half the segment. Take away this triangle from the segment, and from the remaining segments PQ and PQ' take away their corresponding triangles, and from the four remaining segments their corresponding triangles, and so on continually. Thus at length (Euclid x. 1) we arrive at a remainder less than any assignable magnitude.[*] Now the sum of all the above mentioned triangles

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[*] By this continual subtraction the area of the segment is at length *exhausted*. Hence the term "method of exhaustions."

is $\left(1+\frac{1}{4}+\frac{1}{4^2}+\ldots+\frac{1}{4^n}\right)\Delta PQQ'$, and the limit of this when n is infinite is $\frac{4}{3}\Delta PQQ'$, which is accordingly the area of the segment. Notice that at the end of prop. 3 he alludes to an existing treatise on Conics: "These things are proved ἐν τοῖς Κωνικοῖς Στοιχείοις," as he does again in *De Conoid. et Sphæroid.* props. 3, 4. There is no reason to think, as some have done, that he is referring to a treatise of his own.*

The sphere. §4. In the introductions to some of his treatises, Archimedes refers to what had been done by earlier geometers. Thus in the introduction to the *Quadratura Paraboles* (p. 18), having stated as his primary lemma,† that *the excess of one magnitude over another may be continually added to itself till the sum exceeds any assigned magnitude*, he remarks that it had been applied by those before him, viz. to prove that circles and spheres are as the squares and the cubes respectively of their diameters, and that any pyramid or cone (Euclid XII. 7, 10) is the third part of the prism or cylinder having the same base and altitude. In the introduction to *De Sphæra et Cylindro* lib. I. (p. 64), he gives the important information that the cubatures of the pyramid and the cone (Euclid XII. 7, 10) were discovered by Eudoxus.‡ These properties preexisted in the figures, but (though many notable geometers lived before Eudoxus) no one had discovered them. In like manner, none before Archimedes had discovered that the surface of a sphere is equal to four times the area of one of its great circles (prop. 35); the volume of a sphere to two thirds of the circumscribed cylinder having the same altitude, and its surface to two thirds of that of the cylinder (prop. 37); the surface of any segment of a sphere to the area of the circle whose radius is the line from the vertex of the segment to any point on its base (props. 48–9); and the volume of the solid sector determined by any segment to the cone whose base and

Eudoxus B.C. 365.

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* Compare the introductory remarks of Eutocius on Apollonii *Conica* (p. 8, ed Halley).

† See also *De Helicibus* (p. 220).

‡ Eudoxus of Cnidus flourished in the 103rd Olympiad, and died about 357 B.C., according to Bretschneider *Die Geom. vor Eukl.* p. 163.

altitude are severally equal to the surface of the segment and the radius of the sphere (prop. 50).

In lib. II. it is proved that the volume of any segment of a sphere is equal to that of a cone of certain altitude described upon its base (prop. 3); and the theorems now established are applied to solve the problems: to find a plane area equal to the surface of a given sphere (prop. 1), to describe a sphere equal to a given cone or cylinder (prop. 2),[*] and to divide a given sphere into segments whose surfaces or volumes shall be in a given ratio (props. 4, 5).

Circuli dimensio.

§5. In the *Circuli Dimensio* it is shewn, that any circle is equal to the right angled triangle whose sides about the right angle are equal to the radius and the circumference of the circle (prop. 1); a circle is to the square on its diameter as 11 to 14, approximately (prop. 2); and that the circumference exceeds thrice the diameter by a fraction of it less than $\frac{1}{7}$ and greater than $\frac{10}{71}$ (prop. 3). These last results are obtained by regarding the circumference of a circle as intermediate in length to those of its circumscribed and inscribed 96-gons. Thus we see that Archimedes treated the problem both as a quadrature and a rectification of the circle; and he shewed, not only that π is nearly equal to $3\frac{1}{7}$, but that it is *less than* $3\frac{1}{7}$ *and greater* $3\frac{10}{71}$. We may therefore fairly say that his approximation was exact to three places of decimals, since the mean of his two limits gives π equal to 3.1418 &c. The approximation in the Rhind papyrus makes it greater than 3.16 [p. xxvi].

Spiral of Archimedes

§6. In the treatise *De Helicibus* he defines his helix or spiral ($r = a\theta$) as generated by the double motion of a point, which moves uniformly outwards from a fixed origin, in the direction of a radius vector which itself rotates uniformly about that origin. Supposing the generating point to start from the origin O and to arrive at the point A after n complete revolutions of the radius vector, he shews that the intercept made by the tangent at A upon the radius vector at right angles to OA is equal to n times the circumference of the circle described with OA as radius (prop. 19), thus effecting the *rectification* and quadrature of the circle with the help of his spiral; and in prop. 20 he proves the corresponding theorem for any other position OR of the radius vector.* The quadrature of the spiral is determined in props 24–28. From the introduction to this treatise we learn that there were other able geometers in the time of Archimedes, with whom he was in correspondence; and that there were also pretenders addicted to claiming more than their due, for whose discomfiture he propounded false theorems, of which examples are given (p. 218).

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[*] A solution of the problem of the two mean proportionals being here presupposed, Eutocius (pp. 135–149) gives the methods of Plato, Hero, Philo of Byzantium, Apollonius, Diocles, Pappus, Sporus, Menæchmus, Archytas, Eratosthenes and Nicomedes, rejecting that of Eudoxus (pp. 135, 149), perhaps for insufficient reason (Bretschneider *Die Geom. vor Eukl.* p. 166).

His spiral affords the simplest illustration of the generation of curves by an angular compounded with a linear motion; according to the idea of Plato, who "establishing two most simple and principal species of lines, the right and the circular, composes all the rest from the mixture of these" (Proclus on Euclid I. def. 4). Desargues (1639 A.D.) threw out the suggestion that a conic might be thus described, but assigned no law of movement.† On Roberval's rule for drawing the *tangent* to a curve at any point, regarded as the line of the resultant of all the movements of the point, see Chasles' *Aperçu historique* p. 58 (1875).

The conoids. §7. The book *De Conoidibus et Sphæroidibus* contains various theorems on the cubature of the quadrics of revolution, the sphere having been already dealt with in a separate work. The figure generated by the rotation of a "section of the right angled cone" about its axis, that is to say, the paraboloid of revolution, is called the right angled conoid; the hyperboloid of revolution is called the obtuse angled conoid; but the "acute angle conoid," as it should be called, is more briefly termed the spheroid. It is shewn that the areas of ellipses are as the products of their axes (prop. 7); that an infinity of right cones or cylinders can be drawn so as to contain a given ellipse (props. 8–10); and that the plane sections of the conoids and and spheroids are conics (props. 12–15). The book concludes with a series of propositions on their cubature (props. 21–34), which are proved by a process closely related to the method of integration.

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* Thus in effect he determines the trigonometrical tangent of the angle between OR and the tangent to the curve at R.

† Poudra *Œuvres de Desargues* vol I. 227, 298.

Method of exhaustions.

§ 8. The method of exhaustions employed by Euclid and Archimedes involved a tedious *reductio ad absurdum*, and was perhaps first elaborated as a means of *verification* rather than of discovery. The idea of regarding a curve as a limiting form of polygon was propounded, for the case of the circle, by Antipho [p. xxx], in the fifth century B.C.; and the fact that circles are as the squares of their diameters was thus rendered intuitive, presupposing only a well known relation between the areas of similar rectilinear figures. As regards this property of circles and the analogous property of spheres, the proofs given by Euclid may be supposed merely to have established more rigidly what had been already divined by a summary process; but the use of the method of exhaustions was more apparent in the actual evaluation of volumes and areas. Granted, for example, that a curvilinear plane area might be regarded as divided into rectilinear elements by an infinity of consecutive ordinates, the summation of its elements could not well have been effected directly before the invention of some form of algebraical calculus. Instead of regarding the small elements of a curve as ultimately rectilinear, the ancients would (in the case supposed) have proceeded somewhat as follows. Project every ordinate upon the next before and the next after it by parallels to the axis of abscissæ: thus two sets of parallelograms are constructed, to which the area of the curve is intermediate: suppose the difference between the two sets to be indefinitely diminished by increasing the number of ordinates, and then apply the method of *reductio ad absurdum*, as above mentioned. For actual cases of the subdivision of surfaces by parallel planes, and their cubature by this method, see *De Conoid. et Sphæroid.* props. 21–24. The stereometrical work of Archimedes was revived and continued by Kepler, whose *Nova Stereometria* prepared the way for the modern forms of the infinitesimal calculus.

Apollonius B.C. 247–205.

§ 9. Apollonius of Perga was born in the reign of Ptolemy Euergetes (247–222 B.C.), studied at Alexandria under the successors of Euclid, and flourished in the reign of Ptolemy Philopator (222–205 B.C.). Of his various works* the most famous was the Κωνικά, which gained for him (according to Geminus)† the title of the Great Geometer. In the account of this work given by Pappus,‡ it is divided into two tetrads of books, the former founded on Euclid's four books of *Conics*, and the latter supplementary to them. Apollonius in like manner, in his introductory letter to Eudemus, draws a distinction between books 1–4, which he describes as elementary, and the remainder, which were *περιουσιαστικώτερα*, at the same time pointing out that the former also contained very much that was new. The Oxford edition by Halley (1710) contains books 1–4 with the commentary of Eutocius, in Greek and Latin (pp. 1–250); and in a second part, books 5–7 translated from the Arabic and lib. VIII. "restitutus" (pp. 1–171). The volume concludes with the two books of Serenus on the Cylinder and the Cone, in Greek and Latin (pp. 1–88). The contents of the several books of the Conics of Apollonius are specified below. The most striking evidence of his geometrical power is afforded by the fifth book, in which he solves the problem of drawing normals to a conic from an arbitrary point in its plane, and evaluates the coordinates of what we call the Centre of Curvature at any point of a conic. To have worked out such results with the means at his disposal is an achievement not unworthy of the greatest of geometers in any age.

(*a*) Book I. A conical superficies is defined as the surface generated by an infinite right line, which passes through a fixed vertex and moves round the circumference of a given circle: the term Cone is used specially of the finite portion of the superfices between the vertex and the circle or base. The *Axis* is the line from the vertex to the centre of the base. The plane through the axis at right angles* to the base cuts the cone and its base in a triangle, which is called "the triangle through the axis;" and every chord of the cone at right angles to the plane of this triangle is bisected by it (prop. 5). Any plane at right angles to the plane of the triangle through the axis meets the conical surface in general in one of the three curves formerly distinguished as the sections of the right, obtuse and acute angled cones respectively. These names being thus found to be inappropriate, others have to be suggested in their place. The new names may be briefly explained as follows [p. 82]. The Parabola is so called because at every point of it, if p be the parameter, y^2 is *equal* to px; the Hyperbola because y^2 is *greater* than px; and the Ellipse† because y^2 is *less* than px (props. 11–13). He is now practically independent of the cone, and starts afresh from the relation between the ordinate and the abscissa. See also props. 20–21. It is shewn later in the book, that the tangent to a conic at any point and the ordinate of the point to any diameter divide the diameter harmonically (props. 34–38); and lastly a construction is given for describing two conjugate hyperbolas with a given pair of conjugate diameters (prop. 56).

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* See the notices in the *Collectio* of Pappus; and cf. Cantor's *Euclid u. s. Jahrhundert* pp. 44–64.

† Halley's Apollonii *Conica* p. 9. Geminus lived about 150 B.C. (Cantor p. 52).

‡ *Collectio* lib. VII. § 30 (vol II. p. 672, ed. Hultsch).

Application of areas.

In the use of coordinates‡ by the ancients,—as for example by Apollonius in this book, and in a more striking way in his fifth book—the form of procedure was strictly geometrical throughout. Hence we see more clearly the importance of the

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* After prop. 5 this plane is called briefly "a plane through the axis." There is the same laxity of statement in def. 10, where the right line bisecting a system of parallel chords of *any* curve line (πάσης καμπύλης γραμμῆς) in one plane is defined as a diameter; whereupon Eutocius remarks that he rightly adds *in plano*, to exclude the cylindrical helix and the sphere.

† If the three conics were first discovered in the order in which Apollonius (perhaps following Euclid and Aristæus) here introduces them, this tends rather to support the conjecture that they were discovered *in plano* [p. xxxii], since the contemplation of the cone, which was regarded as a finite figure (Euclid XI. def. 18), would have revealed the ellipse first instead of last. Geminus (Proclus on Euclid I. def. 4) called the ellipse θυρεός, from its shape. Cf. cissoid, conchoid, cardioid.

‡ The term *ordinate* was derived by translation from the Greek.

παραβολή of areas [p. xxv.]. In its simplest form this amounted to finding the line which in conjunction with a given line determines a rectangle, or other parallelogram (Euclid I. 44), of given area, which corresponds to the algebraical division of a given product by one of its factors. A further use of the term occurs in the determination of the foci of a central conic which Apollonius calls "the points arising *ἐκ τῆς παραβολῆς*," *puncta ex applicatione facta.* Here the problem is to divide the axis into segments whose product is equal to the fourth part of "the figure" [p. 82], or to determine x and y from the relations $x \pm y = 2a$ and $xy = b^2$. The application to a given line of a parallelogram *deficient* or *exceeding* by a parallelogram similar to a given one is the subject of the propositions Euclid VI. 27–29. For an extension of the method to an indefinite *series* of magnitudes, see Archimedes *De Conoid. et Sphæroid.* prop. 3. Thus the "application" of *areas*, so far as it went, was to the ancient geometry what algebra, which deals with *products* and factors, is to the geometry of Descartes.

(*b*) Book II. The asymptotes are thus defined: on the tangent to a hyperbola at any point P take PT and PT', each equal to the parallel semi-diameter; then the lines CT and CT', and these alone, being produced to infinity, do not meet but approach indefinitely near to the curve (props. 1, 2, 14). Through a given point a hyperbola can be drawn so as to have a given pair of lines for asymptotes (prop. 4). The opposite intercepts made on any straight line by the curve and its asymptotes are equal to one another, and the product of two adjacent intercepts is equal to the square of the parallel semi-diameter (props. 8–11). The product of the distances of any point on the hyperbola from its asymptotes is constant (prop. 12). A line parallel to an asymptote meets the curve in one point only (prop. 13). The tangents to conjugate hyperbolas at the extremities of any two conjugate semi-diameters meet on one or other of the asymptotes (prop. 21). The diameter through the point of concourse of any two tangents to a conic bisects their chord of contact (props. 29–30). Supplemental chords of a hyperbola are parallel to conjugate diameters (prop. 37).

Given a central conic, to find its centre and axes (props. 45–8); and to draw a tangent making a given angle with the axis (prop. 50), or with the diameter through its point of contact (props. 51–3).

(*c*) Book III. The diameters through any two points of a conic contain equal triangles with the tangents at those points (prop. 1). The rectangles contained by the segments of any two intersecting chords of a conic are as the squares of the parallel tangents (props. 16–23). *Any chord through the intersection of two tangents to a conic is cut harmonically** *by their point of concourse and their chord of contact* (props. 37–40): the special case of any chord through the intersection of a tangent and an *asymptote* is treated separately in props. 34–36. Thus a substantial contribution is made to the theory of polars, afterwards completed by Desargues. Any three tangents to a parabola cut one another proportionally (prop. 41). Two tangents being drawn at the extremities of any diameter, the product of their segments by any third tangent is equal to the square of half the conjugate diameter (prop. 42). The tangent to a hyperbola cuts off a constant area from the space between the asymptotes (prop. 43). The foci of a central conic, or "puncta ex applicatione facta," are determined and their principal properties proved in props. 45–52 [p. 111]; but since the process of "application" fails when the axes become infinite, *he does not detect the existence of the focus of the parabola.*

This third book is said by Apollonius, in his preface to the entire work (p. 8), to contain many wonderful theorems, for the most part new; and he adds that Euclid was not able to construct the *Locus ad tres et quatuor lineas* generally,† but only some special case of it, and that indifferently; for in fact it was not possible to complete the construction "without our further discoveries," where the allusion is doubtless to props. 16–23 [p. 266]. From the extant works of Apollonius we learn nothing about the nature of this *Locus*, and even the commentator Eutocius cannot explain the allusion (p. 12). Pappus however informs us what the problem really was, and states clearly that the locus in question is a conic. He then speaks of the locus analogously related to more than four given lines (*Collectio* p. 680, ed. Hultsch). Considering the distances of a point from *six* given lines, we may say that the solid contained by three of the distances varies as that contained by the remainining three; but we cannot go on to more than six given lines, and say: "if the ratio of the content of four of the distances to the content of the remainder be given,—since there is not anything that is contained by more than three dimensions. Nevertheless, men a little before our time have allowed themselves to interpret such things, signifying nothing at all comprehensible, speaking of *the product of the content of such and such lines by the square of this or the content of those.* These things might however be stated and shewn generally by means of compounded proportions &c." These predecessors of Pappus, who were not to be confined to three dimensions, were evidently algebraic geometers, who considered lines not directly as such, but only in their numerical relations to a unit of length.

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* The expression "harmonically" is however not used by Apollonius. On polars with respect to a circle see § 13 (*f*).

† The proof for the case of the circle presents no difficulty [p. 235].

Germ of algebraic geometry.

(*d*) Book IV. No two conics can have more than four points of concourse (props. 25, 36, 40–4, 53), or two of concourse and one of contact (props. 26, 45–8, 54), or two of contact (props. 27, 38, 49–51, 55). Two parabolas can only touch one another in one point (prop. 28).

If the earliest writers on conics had not dealt with the subject of this book (p. 217), and if the principal part of book 3 was also new, we may conclude that the Conics of Euclid contained little or nothing that is not to be found in books 1 and 2 of Apollonius and that the earlier Elements of Aristaeus were meagre, or "somewhat concisely written." Their treatises would have contained elementary propositions on the *right cone*, the relation of the *ordinate* to the abscissa in each section, the property of a *diameter*, some construction of a *tangent* with the determination of its intercept on any diameter, and the leading properties of the *asymptotes*; but nothing about foci, or normals, or the metric relations of conjugate diameters, or of intersecting chords of the general conic drawn arbitrarily.

(*e*) Book v. Under the title *De Maximis et Minimis*, this book treats of the four *Normals* (regarded as greatest or least lines) that can be drawn to a conic from a given point in its plane, and establishes the complete analogue of Euclid III. 7, 8 for the general conic. If P be any point on a conic, N its projection upon the axis, A the nearer vertex, AL parallel to NP and *equal to half the latus rectum*, and Q the point in which the diameter CL meets PN, then

$$PN^2 = 2 \text{ trapezium } ANQL.$$

From this relation between the coordinates PN and AN the following results are elaborated.

If the abscissa AN be less than AL, the least right line from N to the curve is NA, and the greatest is NA' the remainder of the axis (props. 4–6); but if AN be greater than AL, the least line from N to the curve is such that its projection upon the axis is equal to AL in the case of the parabola, and in other cases to $\frac{b^2}{a^2} CN$ (props. 8–10), which is the property of the *subnormal*. The greatest lines from given points on the minor axis of an ellipse to the curve are then considered (props. 16–22), and it is shewn that the intercepts upon them between the curve and the points in which they meet the major axis are the least lines that could be drawn from those points to the curve (prop. 23). All such greatest and least lines meet the conic at right angles (props. 27–33); and if O be any point on one of them and N be the point at which it meets the curve normally, then ON is also a greatest or least line from O to the curve (props. 12, 21, 34). Four *normals* (as we shall now call them) to a semi-ellipse, or three normals to an elliptic quadrant, cannot meet in one point (props. 47–8). If O be any point in the plane of a conic whose abscissa AN is not greater than AL, no normal can be drawn to the conic from O so as to fall within the angle AON (props. 49, 50); but if AN be greater than AL, then according as ON is greater than, equal to, or less than a certain length λ *no* normal, or *one*, or *two* can be drawn to the conic from O so as to fall within the angle AON (props. 51–2).

To determine the length λ, divide the semi-axis CA in H so that

$$CH : NH \triangleq CA : AL;$$

between CH and CA find two mean proportionals* CL and CK, so that

$$CH : CL = CL : CK = CK : CA;$$

and lastly, supposing CK to be an abscissa measured towards A, and P the point on the curve (on the opposite side of the axis to O) having CK for its abscissa, take λ to PK in the compound ratio of CN to CH and HK to CK, so that

$$\lambda : PK = CN.HK : NH.CK.$$

Hence, writing a and b for the semi-axes, we find that

$$CN = \frac{a^2 - b^2}{a^2}.CH = \frac{a^2 - b^2}{a^4}.CK^3;$$

and
$$\frac{\lambda}{PK} = \frac{a^2 - b^2}{b^2}\left(1 - \frac{CH}{CK}\right) = \frac{a^2 - b^2}{b^2}\left(1 - \frac{CK^2}{CA^2}\right),$$

or
$$\lambda = \frac{a^2 - b^2}{b^4} PK^3;$$

and therefore CN and λ are equal to the coordinates of the centre of curvature at P, which is here virtually regarded as the *point of the ultimate intersection of consecutive normals*, since if the ordinate of O be diminished however slightly, O at once becomes a point from which two normals can be drawn. The locus of O is the evolute of the conic [p. 221].

When ON is less than λ, he determines a certain point Σ having CH for its abscissa and a certain point I on PK, and through I he describes one branch of a *rectangular hyperbola* (pp. 39, 40), having for asymptotes the parallels through Σ to the axes of the given conic. This semi-hyperbola intersects the conic in two points X and Y, the normals at which are OX

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* The Arab interpreter gives a construction (p. 40) for two mean proportionals identical with that of Ex. 530 [p. 189].

and OY [p. 265].* To determine the positions of Σ and I, produce ON to a point M such that

$$OM : MN = a^2 : b^2;$$

through M draw a parallel to the transverse axis, meeting the ordinate through H in Σ, and PK in G; and upon PG take the point I, such that $GI . G\Sigma = \Sigma M . MO$. The rectangular hyperbola may then be described. Afterwards he considers other positions of the point O from which the normals are to be drawn, making use of the same hyperbola of construction (props. 55, 58–63). In prop. 75 (the last but two) it is shewn, that if the normals at three points PQR on the same side of the axis of an ellipse cointersect in O, the normal OP nearest to the vertex remote from O is the longest line from O to the semi-ellipse, and the normal OR nearest to the vertex A adjacent to O is the longest line that can be drawn from O to the arc AQ.

(*f*) Book VI. Similar conics being those in which corresponding ordinates and abscissæ are proportional, it is shewn that all parabolas are similar (prop. 11); as also are central conics the figures upon whose axes are similar (props. 12, 13). At the end of the book it is shewn how to cut a section of given form and magnitude from a given right cone (props. 28–30), and conversely, how to draw a right cone similar to a given one through a given conic (props. 31–3).

(*g*) Book VII. In props. 6, 7 use is made of supplemental chords drawn from the vertices. Cf. lib. II. 37. The sum or difference of the squares of conjugate diameters is constant (props. 12, 13); and in the equilateral hyperbola conjugate diameters are equal (prop. 23). The conjugate parallelogram is equal to the rectangle contained by the axes (prop. 31). The relative magnitudes of conjugate diameters in various special cases are then discussed.

(*h*) Book VIII. Of this book there is only a conjectural restitution. Thirty-three propositions are given, containing various special constructions, such as: given the axis and the latus rectum of a central conic, to draw a pair of conjugate diameters whose ratio, or sum, or difference is given (props. 7–12).

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* HALLEY, in a Scholion on Serenus *De Sect. Coni* prop. 39 (p. 69), gives a construction for the three normals to a parabola from a given point, by means of a certain circle through the vertex [p. 224].

Menelaus A.D. 80.

§ 10. In continuation of the account of the most brilliant period of ancient geometry, the century of Euclid, Archimedes and Apollonius, recourse must again be had to the *Collectio* of the much later writer Pappus, for information about the lost three books of Porisms of Euclid. But two other names meanwhile demand at least a passing allusion. In the *Sphærica* of Menelaus, a geometer and astronomer of the first century A.D., is found the theorem (lib. III. lemma 1 p. 83, Oxon. 1758): If the sides *ag*, *gd*, *da* of a plane triangle be met by any transversal in the points *erb* respectively, then

$$ge : ea = gr . db : rd . ba,$$

or *the product of three non-adjacent segments of the sides of the triangle by any transversal is equal to the product of the remaining three.* This was also extended to spherical triangles, and served as a basis for the spherical trigonometry of the ancients, But the property of the six segments *in plano* is here noticed on account of the great results to which it led long after, especially in the hands of Desargues. See also Chasles *Aperçu historique* Note VI. p. 291, 1875; *Les Porismes* p. 107. Menelaus is mentioned in the fourth and sixth books of the *Collectio* of Pappus (pp. 270, 476, 600–2 ed. Hultsch).

Ptolemy A.D. 125—139.

§ 11. Claudius Ptolemæus was "le plus célèbre, sans contredit, mais non le plus véritablement grand astronome de toute l'antiquité." Thus writes Delambre in the *Biographie Universelle* (vol. 36. Paris 1823). In a work on the three dimensions of bodies, Ptolemy introduced the idea of determining the position of a point in space by referring it to three rectangular axes of coordinates (ibid. p. 272). His chief work, which he called a mathematical Σύνταξις, was further described by his admirers as ἡ μεγάλη, and by the Arabs as *Almagest* (ἡ μεγίστη).

In it* (*Dictio Prima*, cap. 12. fol. 9*b*. Venet. 1515) he reproduces the theorem of the six segments (§ 10), which has accordingly been ascribed to Ptolemy—and founds upon it a system of trigonometry, plane and spherical. For a full account of his works see tome II of Delambre's *Hist. de l'Astronomie ancienne* (Paris 1817), comparing the *Præfatio* to the third volume of Hultsch's Pappus.

Pappus A.D. 300.

§ 12. The Συναγωγή of Pappus was formerly best known in the Latin version of Commandinus, but a complete Greek text of its *Reliquiæ* (with a Latin rendering) has at length been edited by Hultsch.† It is a miscellany of mathematics and mathematical history, to which we here refer chiefly to supplement our account of Euclid by some notice of his great work, the lost three books of Porisms. It is customary to place Pappus near the end of the fourth century of our era; but Hultsch, following Usener (*Rheinisches Museum* vol. XXVIII. 403), considers him to have flourished under Diocletian, 284–305 A.D. A general account of the Porisms is given in lib. VII (pp. 636, 648–60), where it is said, that the three books were an exceedingly skilful compilation, serving for the solution of the more difficult problems: the doctrine of porisms was subtle and general, and very delightful to persons of insight and resource: nothing had been added to what Euclid wrote upon them, except that some dull persons had given their second redactions of a few of his propositions. Twenty-nine genera of porisms are specified, and it is stated that the three books contained 38 lemmas and 171 theorems. The 38 lemmas constitute props. 127–164 of lib. VII of the *Collectio* (vol. II. pp. 866–918). Their enunciations are curt and unfinished, being like private memoranda of the writer rather than complete statements, and the whole doctrine of porisms long remained an impenetrable secret. The first great step towards their interpretation was made by Simson (*Phil. Trans.* May 1723), and the most complete work upon the subject is Chasles' *Les trois livres de Porismes d'Euclide rétablis...conformément au sentiment de R. Simson sur la forme des énoncés* (Paris 1860), which contains a historical *résumé*, an analysis (pp. 73–84) and enunciation (pp. 87–98) of the lemmas, and a conjectural restitution of the Porisms.

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* For his property of a trapezium *abgd* inscribed in a circle, viz. $ag.bd = ab.gd + ad.bg$, see cap. 9, fol. 5*b*.

† Pappi Alexandrini *Collectionis quæ supersunt, e libris MSS. edidit &c.* Frid. Hultsch (Berol. 1876–8). Lib. I. is lost, but portions of II.–VIII. remain. The edition is in three volumes, in which the text has one pagination throughout. On earlier editions see Pref. to vol. I.

§ 13. Passing by the lemmas 18, 20, 21, 29–33, 36–38, we may group the remainder as below, adhering (except in one particular) to the classification of Chasles.

(*a*). *Four fixed radiants cut any transversal in a constant cross ratio.*

Lemmas 3, 10, 11, 14, 16, 19. Props. 129, 136, 137, 140, 142, 145.

Lemma 3 (p. 870) affirms, that *if any two straight lines $ABCD$ and $AB'C'D'$ be drawn across three straight lines OB, OC, OD [p. 253], then*

$$AB.DC : AD.BC = AB'.D'C' : AD'.B'C'.$$

Here we have, not quite directly stated, the theorem (*a*). In lemmas 10 and 16 it is shewn conversely, that if $\{ABCD\} = \{AB'C'D'\}$, the line DD' passes through O the intersection of BB' and CC'. Lemma 19 is simply, that if $\{ABCD\} = 1$, then $\{AB'C'D'\} = 1$. Lemmas 11 and 14 follow from 3 and 10 respectively by taking one of the two transversals, as aBb in Art. 103 [p. 251], parallel to one of the three radiants.

These Lemmas are used in the proof of lemmas 12, 13, 15, 17.

(*b*). *The opposite sides and the two diagonals of any quadrilateral meet any transversal in three pairs of points in involution.*

Lemmas 1, 2, 4–7. Props. 127, 128, 130–3.

Lemma 4 (p. 872) will serve as an example of the obscure enunciations of Pappus. The statement is as follows:

The figure $ABCDEFGHKL$, and as $AF.BC$ to $AB.CF$ so let $AF.DE$ be to $AD.EF$. (I say) that the line through the points HGF is straight.

This is in reality a converse of (*b*) [p. 261]. In lemmas 1 and 2 the transversal is parallel to a side of the quadrilateral.

In lemma 5 it is drawn through the points of concourse A and C of the opposite sides of the quadrilateral, and the line AC is divided *harmonically* by the two diagonals, or if parallel to one of them (lemma 6), is bisected by the other.

(*c*). *Theorem reciprocal to the above.*

Lemma 9. Prop. 135.

Lemma 9 (p. 878) is that, if AD and AE be drawn from the vertex A to the base BC of a triangle, FG a parallel to the base meeting AB in F and AC in G, and H a point on DE such that

$$BH : HC = DH : HE,$$

the lines FH, GH meet AB and AC respectively in points K and L lying on a parallel to BC.

This is the converse of a special case of the theorem, that *the three pairs of summits of a quadrilateral $FGKL$ subtend a pencil in involution at any point A in its plane;* the point H in the case supposed being the centre of the involution in which BE and CD are segments.

(*d*). *If a hexagon be inscribed in a line-pair, its three pairs of opposite sides meet in three points lying in a straight line.*

Lemmas 8, 12, 13, 15, 17. Props. 134, 138, 139, 141, 143.

Lemma 13 (p. 886) is to the effect, that if AEB and CFD be triads of points on a straight line, the three intersections

$$(AF, CE), \quad (FB, ED), \quad (BC, DA),$$

are in a straight line. The figure $AFBCED$ may be regarded as a hexagon inscribed in a line-pair. Lemma 12 is the case in which AB and CD are parallel. Lemmas 15 and 17 are converse forms of 12 and 13.

Lemma 8[*] (p. 878) is thus enunciated:

Let $ABCDEFG$ be a βωμίσκος, and let DE be parallel to BC, and EG to BF. Then DF is parallel to CG.

That is to say, if BC be the base of a triangle, DE (terminated by the sides through B and C) a line parallel to BC, and EG, BF a pair of parallels terminated by BD and CE respectively or their complements, then DF, CG are parallel.

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[*] This lemma is isolated in Chasles' classification (p. 78).

In other words, *if FBCGED be a hexagon inscribed in a line-pair BDG, CEF, the intersections (FB, GE), (BC, ED), (CG, DF) are in the case supposed at infinity, and in general in one straight line.*

(*e*). *Harmonic section of a right line.*

Lemmas 22–27, 34. Props. 148–153, 160.

These Lemmas are on the metric relations of the segments of a harmonic range; but the term harmonic, although coined long before [p. xxvi], is not employed.

(*f*). *Property of polars with respect to a circle.*

Lemmas 28, 35. Props. 154, 161.

These Lemmas (pp. 904, 914) are to the effect, that any chord of a circle drawn through a fixed point *without* or *within* it is divided harmonically by the point and a certain fixed straight line. Of this proposition, which in its entirety is the foundation of the theory of polars with respect to a circle, the former part only was extended by Apollonius to the conics (lib. III. 37).

Focus and directrix. § 14. To Pappus we are further indebted for the earliest trace of a focus of the parabola, and of a directrix of any conic. In the *Collectio* lib. VII. prop. 238 (p. 1013) is the theorem, that the locus of a point *in plano*, whose distance from a fixed point varies as its perpendicular distance from a fixed straight line, is a conic. Thus one focus of the parabola is at length found; but it was reserved for Kepler to complete the theory of the real "foci" of conics, and to give them their name.

Serenus. § 15. The two books of Serenus of Antissa *De Sect. Cylindri* and *De Sect. Coni* respectively form a sequel to the Conics of Apollonius in Halley's edition. Serenus was also a commentator on Apollonius, and he lived before Marinus, a disciple of Proclus.* Many geometers in his day imagining that the sections of the cylinder were not identical with the elliptic sections of the cone, he sets to work to remove this misapprehension. (*De Sect. Cyl.* props. 16–18). He then shews how to construct a cylinder and a cone on coplanar bases, so as to be cut by one plane in the same ellipse or in similar ellipses (props. 19–23); and further, given a cylinder and a plane cutting it, he describes a cone having the same base and altitude, which is cut by the given plane in a section similar to that of the cylinder (prop. 25). Props. 26–30 shew how to cut a cylinder or cone in subcontrary pairs of similar ellipses. The remaining propositions *De Sect. Cylindri*, although of still greater interest and importance, are sometimes overlooked. The property of a harmonic pencil, indirectly stated, is applied *in space* to prove that all the tangents to a cone from one point have their points of contact on two generating lines (props. 33, 34), and the idea of projection by rays emitted from a luminous point is suggested and illustrated by a simple case (prop. 35). In the book *De Sect. Coni* he breaks (as he tells us) new ground, in thoroughly discussing the triangular section determined by an arbitrary plane through the vertex. Thus he makes a step towards the generalisation of Desargues, who drew his planes of section without reference to the fixed "triangle through the axis."

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* The date 450 A.D. for Serenus may serve as a conjectural lower limit. Baldi, *Cronica de Matematici* p. 59 (Urbino 1707), boldly assigns to him the precise date 462. Suter *Gesch. der math. Wissenschaften* I. p. 92 (Zürich 1873) prefers the date 200–100 B.C.

Perspective.

§16. The writings of Serenus suggest an answer to the question (Chasles *Aperçu historique* p. 74, 1875), Was the method of perspective* known and used by the ancients? Certainly not by those who doubted whether the sections of a cylinder were also sections of a cone. But Serenus now shews that the property of a harmonic range may be transferred by central projection from plane to plane, and hence that any tangent to a conic section and its point of contact project into a tangent and its point of contact on any plane. The principle of perspective had thus been laid down, as the modern reader clearly sees; but if the ancients had still (as in the time of Apollonius) no complete theory of polars with respect to a conic, and if they had not learned to look upon parallel straight lines as concurrent (Chasles *Les Porismes* p. 104), the method could not have been applied by them to much effect, had it been even more distinctly formulated than by Serenus.

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* For some information about perspective see Poudra's *Histoire de la Perspective ancienne et moderne* (Paris 1864).

fingendus est infinito intervallo a priore remotus, adeo ut educta HG vel IG ex illo cæco foco in quodcunque punctum sectionis G sit axi DK parallelos.*"

In the circle the focus recedes as far as possible from the nearest part of the circumference, in the ellipse somewhat less, in the parabola much less; whilst in the line-pair the "focus," as he still calls it to complete the analogy, falls upon the line itself. Thus in the two extreme cases of the circle and the line-pair the two foci coincide. He then goes on to compare the latus rectum and its intercept on the axis, or as he calls them the *chorda* and *sagitta*, in the several sections, concluding with the case of the line-pair, in which the chord *coincides with its arc*, "abusive sic dicto, cum recta linea sit." But our geometrical expressions must be subject to analogy, "*plurimum namque amo analogias, fidelissimos meos magistros, omnium naturæ arcanorum conscios.*" And especial regard is to be had to these analogies in geometry, since they comprise, in however paradoxical terms, an infinity of cases lying between opposite extremes, "totamque rei alicujus essentiam luculenter ponunt ob oculos." Lastly he shews how to describe an ellipse by means of a string fixed at the foci, without the use of the clumsy compasses [p. 178], "quibus aliqui cudendis admirationem hominum venantur," and gives the corresponding constructions for the hyperbola and the parabola.

Continuity. (1) Hereupon be it remarked, that the principle of Analogy on which he insists so fervently is the archetype of the principle of Continuity. The one term expresses the inner resemblance of contrasted figures A and B, which are connected by innumerable intermediate forms; whilst the other expresses the possibility of passing through those intermediate forms from A to B, without any change *per saltum*. Geometry was *not* indebted to Algebra for the suggestion of the law of continuity.

Second focus of parabola. (2) Having traced the transition from the line-pair to the circle through the three standard forms of conics, he completes the theory of the points henceforth named Foci by the discovery of the "cæcus focus" of the parabola, which is to be taken at infinity on the axis either *without* or *within* the curve. The parabola may therefore be regarded indifferently as a hyperbola, having (relatively to either of its branches) one external and one internal focus, or as an ellipse, having both foci within the curve.

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* The figure indicates that the line from the further focus may be considered to lie either within or without the parabola.

Opposite infinities adjacent.

(3) The further focus of the parabola being taken at infinity on the axis in either direction, the two opposite extremities of every infinite straight line are thus regarded as coincident or consecutive points—a conception which has been shewn to conduct logically to the idea of imaginary points [p. 311].

Parallels concurrent.

(4) Every straight line from the "cæcus focus" of the parabola to a point on the curve being said to be parallel to the axis, the idea of the concurrence of parallel lines at a point at infinity has at length been formed and announced.[*] It is to be noticed that the new doctrine of parallels is here presented in relation to one plane, and not as springing out of the consideration of figures in perspective in space.

Such were Kepler's most original contributions to pure geometry, although he is better known by his continuation of the work of Archimedes in stereometry.

§ 4. *Nova Stereometria doliorum vinariorum.*

Of this work (anno 1615, Lincii. *Op.* IV. 545–646 ed. Frisch) we notice chiefly the former part, which contains a new and abbreviated redaction of the work of Archimedes on the circle and in stereometry, followed by *Supplementum ad Archimedem* (p. 574). The circuitous method of exhaustions is here transformed into the method of infinitesimals. Thus in theor. 1, on the approximation to π, he treats an infinitesimal arc as a straight line: "Licet autem argumentari de *EB* ut de recta, quia vis demonstrationis secat circulum in arcus minimos qui *æquiparantur rectis.*" In theor. 2 in like manner he regards the circle as an aggregation of triangular elements, having a common vertex at the centre, and their bases coincident with successive small arcs of the circumference. So the sphere is considered in theor. 11 to be made up of small cones, having their vertices at the centre and their bases, "quarum vicem sustinent *puncta*," on the surface of the sphere. He then passes on to the conoids &c., and thence to the solids generated in a certain way by conics, the generating curve being attached at right angles to a plane, which turns about one of its own points without change of place. He gives a slight *résumé* of his doctrine of the foci, mentioning the further focus and likewise the centre of the parabola, but not in such a way as bring out the idea of the concurrence of parallels (p. 577).

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[*] It was in the course of an attempt to trace the origin of the term Focus of a conic that I came upon the passage quoted from the *Paralipomena.* Chasles (*Les Porismes* p. 104) attributes the discovery of the concurrence of parallels to Desargues; and when he says (*Aperçu* p. 56. Cf. pp. 15, 16, 61) that Kepler first introduced "l'usage de *l'infini*" into geometry, he is referring "aux méthodes infinitésimales."

§5. Guldinus, quoted by Frisch (Kepleri *Op.* IV. 647), stoutly opposed Kepler's *æquiparatio* of an arc to a chord, as not permissible "per ullam ullius demonstrationis geometricæ vim"; precisely as it was objected to Antipho [p. xxx], who had made bold to do likewise some 2000 years earlier, that "he did not start from geometrical principles." It could not however be denied that Kepler's method was of service in discovering theorems, although by no means to be recommended as a method of proof—at least, if any better could be found, "si alia suppetant geometris jam probata media" (p. 653). Kepler had in reality grasped the idea of the infinitesimal, although a *calculus* remained still to be discovered. The law of continuity is now applied by him not only to the infinitely great but to the infinitely small. He has formed the conception of the continuous change of a variable: "crescit a quantita nulla *continue* &c," and discovered the law of its variation in the passage through a maximum value (Pt. II. theor. 16—22); thus laying a firm foundation for the fluxional calculus of Newton, better known by the name and with the notation proposed by Leibnitz.

Desargues 1593—1662.

§6. The name of Girard Desargues of Lyons (1593—1662) had fallen into oblivion, when early in the present century his genius was recognised by Brianchon (1817) and Poncelet (1822). A further appreciative notice in the *Aperçu historique* of his apparently lost works was followed by Chasles' own discovery of the chief of them, *Brouillon Proiect etc.*, which with others afterwards discovered was published by Poudra in his *Œuvres de Desargues* (Paris 1864). This edition contains a biographical notice of Desargues, his recovered works with an analysis of each, and an analysis of those of his pupil Abraham Bosse, themselves founded upon the ideas of the master. He proclaims himself not in the first instance a pure mathematician, avowing that he had never a taste for study or research except with a view to some practical application, "au bien et commodité de la vie." He was an architect and engineer, and in the latter capacity served under Cardinal Richelieu at the siege of La Rochelle (1628). After the war he retired to Paris, where he devoted himself to geometry and its applications, frequenting a weekly gathering of savants for the discussion of mathematical topics, which preceded the foundation of the *Académie des Sciences* (*Œuvres* I. 14). He was esteemed by the ablest of his contemporaries as a geometer second to none, but virulently attacked by some important persons* of smaller calibre, who were confounded by the novelty and abstraction of his ideas. The subsequent neglect of his works was due partly to the form in which they were written, but in far greater measure to the counter-attraction of the algebraic geometry of his contemporary and friend Descartes. For full information about his works, which include Perspective, Coupe des Pierres, Gnomonique, a fragment on gravitation (I. 239) &c., we can only refer to Poudra's excellent edition; but it will be seen from the following slight account of some portions of them that the *Géométrie Projective* of the present day is in fact the geometry of Desargues.

§7. In his earliest work, *Méthode Universelle de mettre en Perspective &c.* (1636), he notices the cases in which concurrent lines are seen as parallels on the *tableau* (tome I. pp. 83, 94),

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* The hostile critique of M. de Beaugrand, secrétaire du Roi, "est le premier écrit qui a servi au général Poncelet à reconnaître le mérite de Desargues" (*Œuvres* II. 353).

and concludes with the problem, to find the lines in a conic which correspond to the axes of its projection. Purposing to return to this work shortly, we pass on to the BROUILLON PROIECT *etc.* [p. 261], or *rough sketch* of a theory of the intersection of a cone by a plane (pp. 97–242, and 243–302). He commences with the new doctrine of infinity.* The opposite points at infinity on a right line are coincident, parallel lines meet at a point at infinity, and parallel planes on a line at infinity (pp. 103–6, 229, 245–6). A straight line may be regarded as a circle whose centre is at infinity (pp. 108, 224). The theorem of the six segments found in the Almagest and elsewhere is stated in a converse form (p. 256). The theory of *Involution de six points*, with its special cases, is fully laid down, and the projective property of pencils in involution is established† (pp. 246–61). The theory of polar lines is expounded, and its analogue in space suggested (pp. 263–6, 271–7, 214, 291). A tangent is a limiting case of a secant (pp. 262, 274, 277), and an asymptote is a tangent at infinity (pp. 197, 210). The joins of four points in a plane determine three couples in involution on any transversal (p. 266), and any conic through the four points determines a couple in involution with any two of the former (p. 267). The points of concourse of the diagonals and the two pairs of opposite sides of any quadrilateral inscribed in a conic are a conjugate triad with respect to the conic, and when one of the three points is at infinity its polar is a diameter (pp. 188–9); but he does not explain the case in which the quadrilateral is a parallelogram, although he had formed the conception of a straight line wholly at infinity (p. 265).

"Mais voicy dans une proposition comme un assemblage abregé de tout ce qui precède" (pp. 195, 277). Thus he introduces the general theory of projection, which is the main subject of the *Brouillon*. Given any conic O and a cone through it, let O' be any section of the cone. Through the vertex V

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* He must have been acquainted with Kepler's theory of the foci. Notice his use of Kepler's term "foyers" (pp. 210, 222).

† Notice his form of expression (p. 104), "*Rangée* de points alignez" [p. 249].

[p. 314] draw a plane parallel to that of *O′* meeting the plane of *O* in *ab*, and let *P* be the pole of *ab* with respect to *O*. Then the system of planes through *VP* determine the *diameters* of *O′*, the centre being considered, as we should say, to be the pole of the line at infinity; and any two such planes drawn through points on *ab* conjugate with respect to *O* determine *conjugate diameters* of *O′*, the tangents at the extremities of which, "à distance ou finie ou infinie" are also known (p. 197). Since a point at which *O* meets *ab* coincides with its own conjugate, an asymptote of *O′* (besides being a tangent at infinity) is a double or self-conjugate diameter. He concludes: "Comme entr' autre, que sur la quelconque de ces coupes de rouleau peut estre construit un rouleau qui sera coupé selon quelconque espèce de coupe donnée" (p. 198). The ancients had always taken a circle for the base of their cone, and had drawn all planes of section at right angles to one and the same fixed plane.

The Foci of a conic are determined *in plano* as the intersections of the axis with a certain circle, which may have for diameter the intercept on any tangent (or on an asymptote, p. 288) by the tangents at the vertices, in accordance with Apollonii *Conica* III. 45 [p. 111]. He determines the axes and foci of a conic in the cone by a process (pp. 215–23, 293) which Chasles summarises as follows:*

The line *ab* being drawn as above, take any point *t* upon it, and let the chord of contact of the tangents from *t* to *O* meet *ab* in *t′*. Also let *rr′* be any segment of *ab* which subtends a right angle at *V*. The two sets of points *tt′* and *rr′* constitute two involutions, having one segment *cc′* in common. The polars *X*, *X′* of the points *c*, *c′* correspond to the AXES of *O′*. The tangents to *O* from the points *r* and the lines from *r′* to their several points of contact determine upon *X* an involution, whose double points correspond to the FOCI of *O′*, since every tangent and its normal are harmonic conjugates to the focal distances of the point of contact.

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* *Rapport sur les progrès de la Géométrie* p. 305.

§ 8. A sequel to Desargues' *Perspective* of 1636, found in the *Perspective* of Bosse (1648), contains some explanations of the principles of the former work.

a. Proposition fondamentale de la pratique de la Perspective.

The statement and proof by Desargues (*Œuvres* I. 403—7) are analysed by the editor, who reduces his fundamental proposition to the anharmonic property of a pencil of four rays (p. 425), which cut any transversal in a constant cross ratio.

b. Figures in homology.

Three other geometrical propositions are given, which embody the principle of perspective in one plane (pp. 413—22, 430—5). The first is on triangles in homology [p. 307, 321], the second and third on quadrilaterals in homology. On the second he remarks generally that a like reduction of the figure to one plane may be used "en semblable cas" (p. 417); and in the third he gives a metrical relation between a system of corresponding segments (p. 435). Notice that he passes from solid to plane figures in the manner afterwards used by the school of Monge (Chasles' *Aperçu historique* p. 87).

School of Desargues.

§ 9. Although the roughly sketched essays of Desargues themselves fell into neglect, his ideas were preserved by an illustrious school of disciples, numbering amongst its members Bosse, Pascal and De la Hire. The writings of the engraver Abraham Bosse (1643—1667) are analysed by Poudra in vol. II. of the *Œuvres de Desargues.* The famous Hexagrammum Mysticum of Pascal was a corollary from what he had learned from Desargues. The theory of polars was brought into prominence by De la Hire (1685), and forthwith supposed to have been discovered by him. The reader of Brianchon's *Mémoire sur les Lignes du Second Ordre* (Paris 1817), and Poncelet's *Traité des Propriétés Projectives* will not need to be reminded how great a part of modern geometry is actually and confessedly founded on the work of Desargues.

Newton 1642—1727.

§ 10. Newton was born at Woolsthorpe near Grantham in 1642, the year of the death of Galileo, and died in the eighty-fifth year of his age in 1727. The first edition of his *Philosophiæ* *naturalis Principia mathematica* was published 1687, the second, edited by Roger Cotes, in 1713, and the third, by Henry Pemberton in 1726. His *Opticks* was published in English in 1704 and in Latin in 1706, in each case with an *Appendix* in Latin containing the *Enumeratio Linearum Tertii Ordinis* and a tract *De Quadratura Curvarum*. For an account of his life and writings see Brewster's *Memoirs of the Life etc. of Sir Isaac Newton* (Edinburgh 1855), and Edleston's *Correspondence of Sir Isaac Newton and Professor Cotes* (London 1850); and for the works themselves see Horsley's five volumes, 1779–85 [p. 264], and the *Newtoni Opuscula* (3 vols.) of Joh. Castillioneus (Laus. et Genevæ 1744). Presupposing a general acquaintance with his geometrical discoveries, we shall confine our attention to a few particulars.

§ 11. In the fourth and fifth sections of the first book of the *Principia* he solves various forms of the problem, to describe a conic subject to the equivalent of five conditions, (1) when a focus is given, and (2) when neither focus is given. It will suffice to allude briefly in passing to the former case. The title of lib. I. sect. 4 is *De inventione orbium ellipticorum, parabolicorum & hyperbolicorum ex umbilico dato.* In it he makes much use of the simple property (lemma 15), that the perpendicular from one focus of a conic to any tangent intercepts a length equal to the axis on the further focal distance of the point of contact. The section concludes with the construction of an orbit of which one focus and three points are given, a problem which had been solved, "Methodo haud multum dissimili" by de la Hire, *Sect. Conic.* lib. VIII. prop. 25. In this construction and in prop. 20 Newton assumes that the focal distance of a point on a fixed conic varies as its distance from the directrix, a theorem proved in the *Arithmetica Universalis* prob. 24 (Cantab. 1707), and sometimes attributed to Newton as its first discoverer, although in reality known to Pappus.

§ 12. *Inventio orbium ubi umbilicus neuter datur.*

A.

The 5th section of the first book of the *Principia*, under the above title, treats with the utmost generality of the *point-*

properties and the *tangent-properties* of conics. It commences with the problem of the *Locus ad quatuor lineas* (lemmas 17—19), of which no geometrical solution was extant [p. 266]. Then follows a theorem (lemma 20) which may be thus stated: If AB PC be four fixed points on a given conic, the chords from B and C to a variable point on the curve meet the parallels through P to AB and AC respectively in points T and R, such that PT varies as PR, and conversely. From a limiting case of this lemma he deduces his organic description of a conic by means of two rotating angles* (lemma 21), giving somewhat later (prop. 27 Scholium) his construction for the centre and asymptotes of the conic thus generated. By means of the above mentioned lemmas he shews how to describe a conic when five points on it are given,† or four points and a tangent, or three points and two tangents (props. 22—4). Next follows lemma 22, *Figuras in alias ejusdem generis mutare*, in which it is shewn that any curve may be transformed into another of the same order by substitutions of the form $X=\frac{ab}{x}$ and $Y=\frac{ay}{x}$ [p. 330], and two applications of the lemma follow. (1) In order to describe a conic passing through two given points and touching three given lines, he transforms two of the given lines into parallels, and the third given line and the join of the given points into parallels (prop. 25); and (2) to describe a conic passing through a given point and touching four given lines, he transforms the four lines into the sides of a parallelogram (prop. 26).

B.

The lemmas next following lead up to some important properties of the tangents to conics, the discovery of which by Newton is commonly overlooked. First it is shewn that if AC and BD be lines given in position, terminated at A and B, and having a given ratio to one another, the locus of the point which divides CD in a given ratio is a straight line‡ (lemma 23).

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* Cf. Ex. 853 [p. 358]. The equation to the locus in lemma 21 is given in the *Arithmetica Universalis* prob. 53 (Cantab. 1707).

† A solution of this case is found in Pappi *Collectio* (p. 1077 ed. Hultsch).

‡ It also divides AB in the same ratio, since AC and BD vanish together.

Next, if two given parallel tangents viz. at A and B to a conic be cut by any third tangent in M and I respectively, the semidiameter parallel to the two tangents is a mean proportional to AM and BI (lemma 24). From this, which is identical with lib. III. prop. 42 of the *Conics* of Apollonius, he deduces two corollaries: (1) if any fourth tangent meet AM, BI, MI in F, Q, E respectively, then

$$AM : BQ = AF : BI = MF : IQ = ME : EI;$$

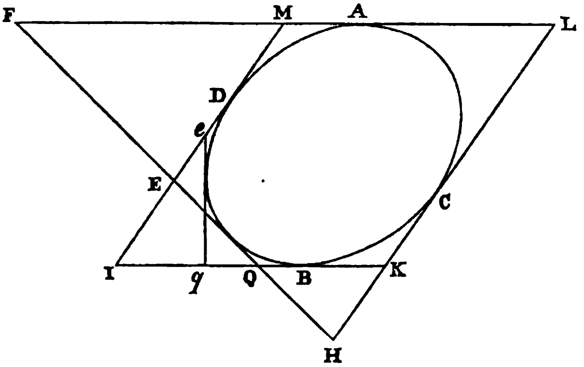
from the first of which proportions it follows (2) that FI and MQ intersect upon AB.* The next lemma and its corollaries are of peculiar importance in relation to the modern geometry.

LEMMA XXV.

If ML, LK, KI, IM be the sides of a parallelogram touching a conic in A, C, B, D respectively, and if any fifth tangent cut them in F, H, Q, E respectively, then by lemma 24 cor. 1,

$$ME : EI = AM : BQ = BK : BQ;$$

or
$$ME : MI = BK : BK + BQ = BK : KQ.$$



In like manner

$$KH : HL = BK : AF = AM : AF,$$

or
$$KH : KL = AM : AF - AM = AM : MF.$$

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* It is easy to generalise this result by transforming the parallel tangents into non-parallels by Newton's method. Cf. Art. 121 Cor. [p. 276].

Corollary 1.

If the parallelogram and the conic be fixed and the fifth tangent variable, then

$$KQ.ME = MI.BK = \text{a constant},$$

and $KH.MF$ has the *same* constant value. The result may be expressed in words as follows: *given two tangents to a fixed conic, the product of the intercepts upon them between the diameter parallel to their chord of contact and any third tangent is constant.* The relation between the intercepts IE and IQ is of the form $(IM - IE)(IK - IQ) = $ a constant, or

$$a.IE.IQ + b.IE + c.IQ + d = 0,$$

which is the "tangential equation" to any conic referred to any two fixed tangents, and also expresses that *any two given tangents are cut homographically by a variable third tangent.*

Corollary 2.

If eq be any other position of the tangent E, it follows that

$$KQ : Me = Kq : ME = Qq : Ee.$$

Corollary 3.

Since QK is to eM as Qq to eE, it follows by lemma 23 that the middle points of KM and qE are in a straight line with the middle point of Qe. Hence, the middle point of KM being the centre of the conic, *if a conic be inscribed in a quadrilateral, its centre lies on the straight line bisecting any pair of the diagonals of the quadrilateral.*[*] This theorem suggested to Brianchon and Poncelet the investigation of the centre-locus of a conic passing through four given points [p. 282], and prepared the way for the general consideration of *systems of conics subject to four conditions.*

C.

In prop. 27 the conic touching five given lines is described, its centre being first determined as the point of concourse of the diameters of any two of the five quadrilaterals formed by the given tangents. In a Scholium he remarks that an asymptote is a *tangent at infinity*, and also shews how to determine the axes and foci of a conic described by the organic method of lemma 21. There are also other lemmas which he might have used for the construction of conics, such as that the locus of the middle point of a chord drawn through a fixed point to a conic is a parallel conic: "sed propero ad magis utilia." In lemma 26 and a corollary it is shewn how to describe a triangle of given species and magnitude having its vertices severally on three given lines, and how to draw a transversal the intercepts upon which by three given lines shall be of given lengths. This lemma is used in prop. 28. In lemma 27 and a corollary it is shewn how to describe a trapezium of given species having its vertices severally on four given lines, and how to draw a transversal which shall be cut in given ratios by four given lines, which is a special case of section in a constant *cross ratio* [p. 296]. An application follows in prop. 29, and the section concludes: "Hactenus de orbibus inveniendis. Superest ut motus corporum in orbibus inventis determinemus."

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[*] It may or may not have occurred to Newton that this theorem might be generalised by projection; but in any case he would not have turned aside to notice results so distantly related to his *Inventio orbium.* It may be shewn that he must have been acquainted with the theory of Polars.

§ 13. *Curvarum Descriptio Organica.*

A well known generalisation of Newton's description of a conic by angles would certainly have been passed by in the *Principia* with a "propero ad magis utilia," since it merely shews how to describe a conic by assuming that a conic is already described. When however he is treating of pure mathematics, he extends his method to the utmost, applying it not merely to cubics, as in Ex. 760, but to curves of all orders having "double" points. In the case of a cubic, of which a *double point* A and six other points $BCDEFG$ are given, let the angle CAB turn about A and the angle ABC about B; then as the intersection C of the arms AC, BC assumes the new positions $DEFG$, the intersection I of the other arms determines four other points, say $PQRS$. Draw the conic $APQRS$, and let I move round its circumference: then C traces the cubic as required (*Opticks*, II. 160, 1704).

If instead of the point C a *tangent* BC be given, the angle CAB vanishes, and the curve is described by means of one finite angle and a straight line, which latter moves parallel to itself when the fixed point through which it passes is at infinity.* How was Newton led to his organic description of conics and other curves? Possibly he took a suggestion from Euclid III. 21 [p. 172], and first described a *circle* by means of an angle and a line parallel to one of its arms, or by two angles having one pair of their arms constantly parallel.

§ 14. *Proof and extension of Newton's Descriptio Organica.*

Let two angles AOB and $A\omega B$ of given magnitudes turn about O and ω respectively, and let the intersection A trace a curve of the nth order. For a given position of the arm OB there are n positions of A and therefore n of B. When OB is in the position $O\omega$ the n B's coincide with ω, which is therefore an n-fold point on the locus of B, as is also the point O; and since any line through O (or ω) meets the locus of B in n other points, the locus is of the order $2n$. Its order is the same when $A\omega B$ is a zero-angle or straight line.

Let a given trihedral angle O (ABC)—or a plane OBC and a line OA rigidly attached to it—turn about O, and let a variable plane through a fixed point ω meet OA in A and the plane OBC in BC; then if the line BC describes a ruled surface of the order n the point A describes a surface of the order $4n$.† For a given position of the line BC the locus of A is a conic, and when the director surface is a cone of the nth order every plane through ω and its vertex meets the surface which is the locus of A in n conics.

When the director is a plane, BC must be made to pass through a fixed point or touch some curve in it, *except* in the case in which OA is normal to the plane OBC. In the last case the locus of A is a quadric, which becomes a sphere when the director is at infinity.

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* For some earlier essays at Descriptio Organica see Chasles' *Aperçu* p. 100.

† For the determination of the order of the surface described by the point A I am indebted to Professor Cayley.

Maclaurin 1698—1746.

§ 15. Colin Mac Laurin developed Newton's theory of curves in his *Geometrica Organica, sive descriptio curvarum universalis* [p. 345]. He also wrote a treatise on *Fluxions*[*] (Edinb. 1742). The work *Algebra with an Appendix etc* [p. 128] was first published, after the death of the author, in 1748, and in the same year *An Account of Sir Isaac Newton's Philosophical Discoveries.* The *Appendix* to the Algebra (pref. p. xi) was founded on Cotes' theorem of harmonic means, of which further use has been made by Poncelet and other modern geometers (Salmon's *Higher Plane Curves* § 132 1879).

Ratio determinans.

§ 16. The property of the focus, directrix and determining ratio remaining buried in the *Collectio* of Pappus, modern writers looked to later works for the first notice of the focus of the parabola and the directrix of the general conic. Thus Robertson in his *Sect. Conic.* pp. 340, 363 (Oxon. 1792) refers for the focus of the parabola to an anonymous work *De Speculo Ustorio*—known to Roger Bacon, and perhaps translated from the Arabic—which was published at Louvain in 1548. Gregory St. Vincent knew the property of the directrix for the case of the parabola, and virtually arrived at it for the ellipse (*Opus Geomet.* lib. IV. prop. 139, p. 317, 1647), in the form of Ex. 16 [p. 35]; as did De la Hire for the hyperbola, measuring distances from the directrix parallel to an asymptote [p. 155 Ex. 397], in his *Sect. Conic* lib. VIII. prop. 18 (1685). In prop. 25 De la Hire introduces the directrix as the polar of the focus: in props. 23–4 he had proved that the tangents at the extremities of any focal or other chord subtend equal angles at the focus. It remained for Newton to bring the property of the determining ratio fully to light, and for Boscovich, with the help of it, to compose the first really complete *Sectionum Conicarum Elementa.* The term Directrix, formerly used in the case of the parabola, is now used of the general conic (p. viii.); as in the following year also (1758) by Hugh Hamilton, who added a construction for the focus and directrix in the Cone [p. 204]; but the term was still used only for the parabola by Le Seur and Jacquier (1760) in their edit. 2 of the *Principia* (I. 134, 179), and the characteristic property of the line itself in the general conic was not familiarly known even some years later, to judge from Lexell's elaborate discovery of the simplest of corollaries from it in the *Nova Acta Petropol.* I. (147), 1787.

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[*] On the rival claim of Leibnitz to the first discovery of the differential calculus see Montucla *Hist. des Math.* vol. II. 330—343 (1758); Gerhardt's *Hist. et Origo Calc. Diff.* (Hannover 1846) and other publications; Brewster's *Life of Newton* II. 23—47 (1855); Weissenborn *Die Principien der höheren Analysis* (Halle 1859); Sloman *The claim of Leibnitz to the invention of the Differential Calculus, translated from the German* (Cambridge 1860). It is disputed to what extent Leibnitz was indebted to the letters and MSS. of Newton. Leibnitz several times discovered things already in print (*Biographie Universelle* XXIII. 627—8, 634); and it is a striking fact that the leading propositions of the PRINCIPIA reappeared under the name of G. G. L. in the *Acta Eruditorum* pp. 82—96 (cf. p. 86), 1689.

Later works founded upon the properties of the determining ratio and the eccentric circle [p. 3] were those of Thos. Newton (Camb. 1794), G. Walker of Nottingham (Lond. 1794), and John Leslie (Edinb. 1821), who describes Thos. Newton's work as "clear, neat, and concise," whilst Walker's "though ingenious and strictly geometrical, is unfortunately so prolix and ponderous as to damp the ardour of the most resolute student." Walker was under the impression that he was the first discoverer of what he called the Generating Circle,* but Thos. Newton rightly referred it to Boscovich. Leslie's account of the work of Boscovich† is that it consisted "of only a few propositions, but drawn out into a string of corollaries." It is nevertheless a clear and compact treatise, which for simplicity, depth and suggestiveness will not readily be surpassed.

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* Compare Mr. S. A. Renshaw's *The Cone and its Sections* pp. 26–8 (Lond. 1875); *Messenger of Mathematics* vol. II. 97 (1873). Walker's complete treatise was to be in five books, but one only appeared.

† From the preface to vol. III. of the *Elementa Univ. Math.* (1757) we gather that Boscovich's scheme of Conics was first published "in Romano litteratorum diario ad annum 1746," in the form of a short article, "schediasma brevissimum"; and that several years elapsed before the complete work, after repeated delays, was given to the world. Leslie says that it was published in 1744, and Walker's in 1795. Walker had discovered the generating circle "near thirty years" before publication. An edition of Boscovich's *Elementa* seems to have been published at Rome in his name in 1752—3; but I have only seen the "editio prima Veneta."

SECTION IV.

MODERN GEOMETRY.

§ 1. ALTHOUGH the law of Continuity, the vital principle of the modern geometry, had been decisively laid down by Kepler, it was not until the great discovery by Poncelet of the circular points at infinity in any plane that it came to be universally acknowledged. The principle as enunciated by Kepler was wholly independent of algebraical considerations, but its later developments were suggested by the occurrence of negative and imaginary roots in equations applied to geometry, whilst the discovery of the differential calculus gave a new zest to speculations *De infinito* and *De nihilo.* The earliest thorough and geometrical treatment of the subject with which we are acquainted is found in Boscovich's appendix* to his *Sectionum Conicarum Elementa* [p. 311], of which a slight account is given below. The complete dissertation occupies more than two-thirds as much space as is devoted to the entire subject of conics *in plano.* The writer cautiously abstains from the too bold assertion of novelty in his speculations, but remarks that the essay contains many things which "ego quidem nusquam alibi offendi," and many which, although found elsewhere, "nusquam ego quidem ad certos reperi redacta canones, et geometrica methodo pertractata." **Continuity.**

§ 2. His first principle is, that every member of a geometrically defined locus must have the same nature and properties,† **Boscovich 1711—1787.**

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* *Elementa Univ. Math.* tom. III. 228—356 (Venet. 1757).

† See Chasles' *Aperçu historique* p. 197 on Monge's use of the principle of *contingent* relations.

which are wrapped up in the definition itself, so that whatsoever is demonstrable of one part of the locus should be demonstrable in like manner of every other. On this principle we conclude *a priori* from the nature of the problem, to trisect a given circular arc, that any construction must give a series of solutions, three of them geometrically distinct [p. 141]. In geometrical demonstrations a determinate configuration is present to the senses, but the reasoning applies to an infinity of cases. This is clearly seen when, for example, we bisect a given finite straight line; but it is true none the less in cases in which a new configuration seems to render the proof nugatory, although some artifice may be required, "ad servandam analogiam, et retinendam solutionis ac demonstrationis vim" (p. 229). Notice his use of the term Analogy, by which the idea of geometrical continuity was first expressed.

The correspondence of change of sign with change of direction in lines carries with it the idea of negative rectangles and squares, and thus of imaginary magnitudes (pp. 234, 308). Change of sign implies a transition through *zero* or *infinity*, and never takes place *per saltum* (p. 250). To illustrate this, take an indefinite line AB and a point C without it, draw CH perpendicular to AB, and let a line turning continuously about C meet AB in P. As CP passes through H, the sign of HP changes, say from positive to negative: when CP becomes parallel to AB, the point P is at infinity on the *negative* side of H, and the next instant it is at infinity on the positive side of H. Thus the passage through infinity carries with it a change of sign and, like the passage through zero, is effected by the *continuous* rotation of CP, and not *per saltum*. The opposite extremities of an infinite straight line are thus in a manner joined, as if the line were an infinite circle (p. 254), whose centre may be considered to be at infinity on either side of the line. In illustration of the principle that opposite infinities are thus adjacent, take the case of an infinite double ordinate to the axis or any diameter of the parabola, regarded as a closed curve (pp. 265, 343).

The consideration of a circle of infinite radius leads to the idea of a "veluti plus quam infinita extensio" (pp. xv, 281).

Through fixed points AC on an indefinite straight line $MACN$ draw a circle, and bisect its minor segment AIC in I and its major segment in I'. If now the centre be removed to infinity, the arc AIC becomes the line AC, whilst of the arc $AI'C$ part coalesces with the infinite segments AM and CN, and the rest recedes to infinity with the point I', "ut nusquam jam sit;" or as we may say, the circle degenerates into the endless line $MACN$ together with the line at infinity* [p. 344]. Hence it is deduced that whilst the line AC is bisected in a point I, its complement $AM \infty NC$ is bisected at a point I' at infinity (p. 274); which might also have been arrived at by dividing AC harmonically, and making one point of section coalesce with the middle point of AC (pp. 6, 344).

If AC be a segment of an infinite right line, the remainder $A \infty C$ may be regarded as its "complementum ad quendam veluti infinitum circulum" (pp. 276, 280, 292, 297 &c.). The hyperbola, regarded as a quasi-ellipse, has for its axis $A \infty A'$ the complement of AA' (pp. 264, 276, 289, 292). The further developments of this idea already given [pp. 18, 22, 102, 153, 311] are in accordance with the views of Boscovich; but the note on Art. 13 was written, and a Scholium in continuation planned [p. 102], before his dissertation on continuity had been consulted.†

The change from the real to the imaginary state is contingent upon the transition of some element of a figure through zero or infinity, and never takes place *per saltum* (p. 277). Examples of imaginaries are the exterior diameters of a hyperbola, whose squares are negative; for the so-called "secondary" axis and diameters have no real analogy to the minor axis and conjugate diameters in the ellipse, although the unwary geômeter may be imposed upon by the conjugate hyperbola, and persuaded to think that there can be a curve of the second order which a straight line meets in four points (pp. 311–317). In comparing properties related to diameters in the ellipse and in the hyperbola, we should endeavour to bring in the *squares* of the diameters, the signs of certain of which will merely have to be changed in passing from the one curve to the other (p. 320). The general method of procedure in dealing with a geometrical figure one or more of whose elements is evanescent, infinite or imaginary is summed up in eleven CANONS, formally stated and fully illustrated (pp. 284–339). The 5th relates to negative angles, and the 11th to the ratios of infinite magnitudes. He might have added a Canon 12 on the ratios of the Newtonian nascent or evanescent quantities, but promises another volume when time permits (p. 348); whilst on p. 353 he refers to his former dissertation *De Natura et usu infinitorum et infinite parvorum* (1741). In the course of the essay now under consideration he treats of curves of all orders, their infinite branches and asymptotes, their curvature, cusps, points of inflexion, and the tangents thereat (pp. 245, 267, 270–3, 325).

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* Although the arc $AI'C$ seems to lie wholly on one side of the line AB, it is to be remembered that *opposite infinities are adjacent.* Thus every line-circle passes through all points at infinity in its plane.

† Without reference to the idea of an infinite line-circle, I had used the term COMPLEMENT of a straight line several years before I was acquainted with any work of Boscovich. See *Oxf. Camb. Dubl. Messenger of Mathematics* IV. 140 (1867).

§ 3. The general solution of the problem, *given the focus directrix and determining ratio, to find the intersections of an arbitrary line with the conic*, completely determines by implication the nature and properties of the curve (p. 286). His construction is as follows. Take a point ω exterior to the given line, instead of a point O upon it [p. 10]; draw ωZ parallel to PQ meeting the directrix in Z, and let the parallel through Z to SR meet the eccentric circle of ω in p' and q'. Then the focal radii parallel to $\omega p'$ and $\omega q'$ meet the given line at its intersections with the conic (p. 39). This construction failing (1) when PQ is parallel to the directrix, in which case the line $Zp'q'$ is indeterminate, and (2) when PQ passes through S, in which case the parallels through S to $\omega p'$ and $\omega q'$ coalesce with PQ, instead of cutting it each in one point; he shews how to meet the difficulties thus arising. The former case however leads only to a simplification when the centre of the circle is taken upon PQ, as in the present work. Considering the second case in relation to Art. 16 Cor. 1 [p. 29], let a fixed chord drawn through the focus S be intersected in Σ by a variable chord containing a given angle with the axis. Then the products of the segments of the two chords are as the squares of the parallel tangents, however near Σ be taken to S, and therefore in the limit the products of the segments of any two *focal chords* are as the squares of the parallel tangents.

A line still meets a conic in two points, even though, for example, one of them should disappear at infinity. In the parabola, any two chords as they become infinite are in a ratio of equality. In the hyperbola, if from any two points LL' parallels be drawn meeting the curve in PP' and its adjacent asymptote in hh' respectively, then as the latter points recede to infinity the intercepts Ph and $P'h'$ remain finite [p. 146], and the ratios of LP to Lh and $L'P'$ to $L'h'$ tend to equality as their limit. The infinite segments LP and $L'P'$ are as the distances of L and L' respectively from the asymptote hh' (p. 347).

§ 4. It is remarkable that Boscovich enters upon these abstruse speculations in an elementary treatise for beginners, and even several times touches upon the subject of the appendix in the text itself, as for example when he notices that the properties of chords of a conic may be transferred to one of its limiting forms, the line-pair (p. 100). The preface to the volume contains an earnest plea for the introduction of the modern ideas into the schools. He had taught the appendix *viva voce* to his own tyros with the happiest results. The mind of the tyro is commonly overwhelmed with a multitude of details not reduced to any system; demonstrations are put before him in an unsuggestive form which gives no play to his inventive faculty; and thus it comes to pass that of the many students so few turn out genuine geometers. Let the learner be furnished with principles, and not alone with fully explained facts, and be continually stimulated to exertion by the intense pleasure of finding something left to discover for himself.

§ 5. The newly founded *Ecole Polytechnique* led the way in the geometrical revival of the nineteenth century. From this source issued first the works of Carnot and Monge, which further illustrated the principles of continuity above described. The leading conception of Carnot's *Géométrie de position* (Paris 1803) is the doctrine of quantities "dites positives et negatives" (p. ii), to which he recurs in his *Essai sur la théorie des Transversales* &c. p. (96), Paris 1806. This essay is in great measure based upon the ancient theorem of the six segments [p. l.].

Referring to the *Aperçu historique* for a good description of the works of Monge, we pass from the master to one of his most illustrious scholars, whose short incisive essays in pursuance of the ideas of Desargues, Pascal and Newton were the prelude to their fuller development by Poncelet, Steiner and Chasles.

Brianchon.

§ 6. Second in importance only to the principle of Continuity is the principle of *Duality*, of which Brianchon's hexagram (1806) occasioned the discovery [p. 290]. Noticing again the important article of Brianchon and Poncelet on the Equilateral Hyperbola [pp. 175, 282], we next come to the separate publications:*

(1) *Mémoire sur les lignes du Second Ordre, faisant suite aux recherches publiées dans les journaux de l'Ecole royale polytechnique.* Par C. J. Brianchon, capitaine d'artillerie, ancien élève de l'Ecole Polytechnique. Paris 1817.

(2) *Application de la théorie des Transversales* (Brianchon. Paris 1818).

The latter memoir consisted of *Leçons données à l'école d'artillerie de la garde royale en mars* 1818. The former, which is of greater interest, must be described in detail. A line of the *Second Order* is defined as the section of any circular cone by an arbitrary plane: the term *projection* is introduced in relation to perspective: *poles* and *polars* are defined: as also is the expression *Géométrie de la règle.* The term polar had been introduced by Gergonne as correlative to "pole," an old expression for a fixed point, which was beginning to be used in its restricted modern sense (Gergonne's *Annales* I. 337; III. 297).

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* It would be worth while to republish Brianchon's articles and memoirs in one volume.

Pp. 7–10. The property (afterwards called anharmonic) of four radiants is enunciated, the case of the harmonic pencil specially noticed—the term *faisceau harmonique* being introduced apparently as new—and it is noticed that the harmonic property holds "pour toutes les projections de la figure," a reference being given to Gregory St. Vincent's *Opus Geometricum* p. 6, prop. 10 (1647). A fourth harmonic to three given points in a straight line is found "avec la règle seule" by the property of the complete quadrilateral, and a reference for this is given to De la Hire's *Sectiones Conicæ* p. 9, prop. 20 (1685).

Pp. 10–16. Any transversal is cut in involution (1) by the sides and diagonals of a quadrilateral, regarded as the *projection of a parallelogram:* and (2) by these and any circumscribed conic regarded as the *projection of a circle.* The latter theorem (§ XI) was due to Desargues and had been preserved by Pascal. The case in which a conic degenerates into a line-pair is noticed. In a note (p. 14) he refers, on the theory of transversals, to the works of the ancients—the *Almagest* for example; to Fr. Maurolycus *Opuscula Mathematica* p. 281, 1575; and to Schubert in the *Nova Acta Petropol.* tome xii. ann. 1794.

Pp. 17—28. Pascal's theorem is proved by considering the six points of concourse of a conic with any triangle, and in a note is added the property of triangles in homology. The extension of the theory of polars to quadrics is ascribed to Monge (p. 19), although in reality due to Desargues [p. 329], who would however have thought definitely only of the quadrics of revolution. The theorem that the joins of four points on a conic determine a self-polar triangle with respect to it [p. lxii] is proved, and the reciprocal property of four tangents deduced. Hence follows "une propriété bien remarquable des lignes du second ordre," viz. that *the intercepts on any two fixed tangents by the diameter MN parallel to their chord of contact and a variable tangent EI have their product EM.NI constant* (§ XXVIII). He does not seem to be aware that this is one of Newton's theorems [p. lxviii] although he refers to Newton more than once (pp. 38, 45), has a general acquaintance with lib. I. sect 5 of the *Principia* from which it is taken, and knows the third corollary of the lemma to which it is the first corollary.

Pp. 28—53. He draws conics passing through n points (one or more of which may be at infinity) and touching $5-n$ lines (where n is 0, 1, 2, 3, 4 or 5), referring also to Newton's methods, and in one case to Maclaurin's *Algebra*. He makes use in these constructions of his own property of the hexagram, the reciprocal of Pascal's; and from the two together deduces (p. 35), that the six summits of two triangles touching a conic lie on a conic, and conversely.

Pp. 53—60. The theorem of Desargues (§ XI) "va nous découvrir de nouvelles propriétés des coniques à branches infinies." Take four points $UXYZ$ on a conic, of which XY are at infinity and U variable, then any fixed chord AB is met by UX and UY in points C and F, such that the cross ratio $\frac{AC}{AF}:\frac{BC}{BF}$ is constant (p. 54). If $AXYZ$ be fixed points on a parabola or hyperbola, of which Z only is at infinity, and U a variable point on the curve, the lines UX and UY meet AZ at distances from A which are in a constant ratio. By making U coincide with each of the points XY, he deduces Ex. 429 [p. 159] and its analogue for the parabola; as also Ex. 427, that the arms of any angle in a fixed segment of a hyperbola intercept a constant length on either asymptote. By means of these results he shews how to describe a hyperbola, having given an asymptote, and in addition three points or a point and two tangents or two points and one tangent. All that remained to bring the *anharmonic point-property of conics* fully to light was a simple application of the method of projection, which the writer had already used with such effect. Knowing so well the importance of the projective property of the anharmonic pencil, it is remarkable that he should have left it for others to take the final step.

"Toutes les proportions contenues dans ce Mémoire se rattachent au théorème XI" (p. 61). Thus the name of Desargues is brought effectually into notice. He also refers to Lambert (*Perspective* ed. 2, 1774), Blondel, Muller &c.

Pp. 61—65. He deduces from Pascal's hexagon the properties of similar central conics, and concludes with some new properties of the tangent cones to quadrics. If a conic passes through two fixed points AB and touches two given lines, the chord of contact passes through a fixed point on the line AB (p. 20). Hence, if a quadric passes through two fixed points AB and has a given enveloping cone, the plane of contact passes through a fixed point on the line AB. A construction is deduced for a quadric passing through *four given points* and having a *given enveloping cone.*

Poncelet 1788—1867.

§ 7. To Poncelet as a geometer belongs the double honour of supplying what was lacking in the theory of Continuity by his discovery of the focoids [p. 311],* and bringing to light the principle of Duality by his method of reciprocal polars [p. 346]; whilst, like Desargues and Archimedes before him, he was no less a master of the principles and practice of mechanics. Born at Metz on the first of July 1788, he was allowed to grow up almost without instruction at St. Avold, until in his seventeenth year, at the end of 1804, he entered the Lycée imperial de Metz. Three years later he gained his admission to the *Ecole Polytechnique*, was employed in 1811 upon the fortifications of Rammekens in the island of Walcheren, marched in 1812 with Napoleon to Moscow, and was taken prisoner and interned at Saratov on the Volga until the general peace of 1814. In his captivity he set to work, in spite of all hindrances, to reconstruct for himself a course of mathematics, and entered upon those bold speculations which are the characteristic of his famous works on geometry. For a full list of his scientific publications see Didion's *Notice sur la Vie et les Ouvrages du Gen. J. V. Poncelet* (Paris 1869). The appearance of his *Cours de Mécanique appliquée*, dating in part from 1826, is described as having "fait sensation dans le monde de la science et de l'industrie" (p. 33). The dominating idea of his geometrical works was to increase the resources of pure geometry, to generalise its conceptions and language, and thus to raise it to the level of analysis. See the Introduction to his *Traité des Propriétés Projectives des Figures* edit. 1 p. xxxiii. (1822); edit. 2 tome I p. xxii (1865–6). At the end of the year following he died (Dec. 23 1867), with his thoughts turned again to mechanics: "Ma tête est bonne, et j'espère bien pouvoir cet hiver publier ma Mécanique (p. 45)."

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* Poncelet, before Plücker, spoke of a conic as having four foci (*Propriétés Projectives* p. 271, 1822).

Steiner.

§ 8. One of the leading contributors to the further systematisation and development of geometry was Jacob Steiner, the author of numerous mathematical articles and of the works:

(*a*) *Systematische Entwickelung etc.* 1832 [p. 262].

(*b*) *Die geometrischen Konstructionen, ausgeführt mittelst der geraden Linie und eines festen Kreises* (Berlin 1833).

The work (*a*) was a first instalment of a treatise in five parts, of which no further part appeared in the author's lifetime; but his *Vorlesungen über synthetische Geometrie* were edited posthumously by Geiser and Schröter (Leipzig 1867, 1876). Of the second part of the *Vorlesungen*, containing the projective geometry of conics, the third section is on *Kegelschnittbüschel* and *Kegelschnittschaar.*

Chasles 1793–1880.

§ 9. Michel Chasles, the most famous of living* geometers, is perhaps best known as the author of the *Aperçu historique sur l'origine et le développement des Méthodes en Géométrie etc.* (Paris 1837, 1875), which contains an invaluable series of notes on the history of geometry from the earliest times, followed by a *Mémoire de Géométrie* (pp. 573–848) devoted to the exposition of the two general principles of Duality and Homography. Supplementary to the *Aperçu* was his *Rapport sur les progrès de la Géométrie* (1870), forming one of a series of official reports on the various branches of literature and science. He is also author of treatises on the *Géométrie Supérieure* (1852, 1880), *Porismes d'Euclide* (1860), *Sections Coniques, première partie* (1865), and of a multitude of separate articles, several of which relate to Maclaurin's theorem in attractions (*Comptes*

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* These pages were already in type when the death of Chasles took place, on the 18th December, 1880.

Rendus v. 842. vi. 808–812, 902–915. 1837–8, &c.). At this point we may fitly offer some remarks upon the history of the "anharmonic" properties of conics and the general principle of "homography," with which the name of Chasles is so intimately associated.

(*a*) *The anharmonic point-property of conics.*

From the *Collectio* of Pappus we are led to infer that Euclid was acquainted with a form of the theorem (1) that the cross ratios of four fixed radiants are constant, and Apollonius with the theorem (2) that the locus $\alpha\gamma = k.\beta\delta$ is a conic. From the union of these two at once arises the well-known "Propriété anharmonique des points d'une conique," which nevertheless remained unnoticed for upwards of 2000 years longer. Although the theorem (1) was rediscovered by Desargues and taken as his fundamental property in Perspective, whilst (2) was brought into notice by Descartes and afterwards proved synthetically by Newton, the combination of the two was not yet thought of.* The third and last stage in their history was inaugurated by Brianchon, who proved that, *if AB be a fixed chord of a conic and XY its points at infinity, the chords from a variable point on the curve to $ABXY$ cut AB in constant cross ratios.* Chasles shewed, in course of an account of his "Transformation Parabolique," that the same is true when X and Y are any two fixed points on the conic; and he deduced that the locus of the point at which four given points subtend a *harmonic* pencil is a conic through the four points. See Quetelet's *Correspondance Math. et Physique* tome v. 293–4, 301 (1829): Chasles *Rapport etc.* p. 268. All that was still wanting was a familiarity with the "théorie complète des rapports anharmoniques,"† which might have been found in the Barycentrische Calcul of Möbius (1827). The property of conics now under consideration is fully stated, and its importance pointed out in the *Aperçu historique* (pp. 80, 334–341), which was presented to the Brussels Academy in a rudimentary form in 1829, and ultimately published in 1837: it is also found in Steiner's *Systematische Entwickelung*, published in 1832 [p. 262].

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* Although it is convenient to deduce Newton's description of a conic by angles from the four-point property [p. 264], we ought, historically speaking, rather to reverse the process, and say that the anharmonic property is evidently contained in his *Descriptio Organica.*

† See the Preface to Cremona's *Géométrie Projective* p. xv (Paris 1875). In this work, originally written in Italian, the reader will find references to many of the leading treatises and historical facts of geometry.

(*b*). *The anharmonic tangent-property of conics.*

It was shewn by Newton that, if IM and IK be certain fixed segments of two given tangents to a conic, any third tangent cuts them in points E and Q respectively such that the rectangle $(IM-IE)(IK-IQ)$ is of constant magnitude; and the same theorem was reproduced by Brianchon in 1817 [p. lxxix]. Poncelet, who refers to Brianchon's *Mémoire*, proves* that, if the opposite sides AB and CD of a fixed quadrilateral circumscribed to a given conic be met by any fifth tangent in L and N respectively, and if BC, DA be met by any sixth tangent in M and P, then

$$\frac{DP}{AP}\cdot\frac{BM}{CM}=\frac{BL}{AL}\cdot\frac{DN}{CN};$$

and by fixing the tangent MP he deduces that *the cross-ratio* $\frac{BL}{AL}:\frac{CN}{DN}$ *of the segments of AB and CD by any fifth tangent is constant.*† In the case of the parabola this cross ratio is equal to *unity* [p. 295]—a theorem which he believes to be due to Halley (p. 118).‡ Chasles gave a second proof of Poncelet's generalisation, regarding the tangents to a conic as projections of the generators of a ruled hyperboloid, and shewed how to pass from it to Newton's theorem, which however he ascribed only to Brianchon (Quetelet's *Correspondance* IV. 364–70. Cf. Chasles *Rapport* p. 239). He afterwards proved it again in the form, that the ratio of the products of the distances of the fifth tangent from A, C and B, D respectively is constant (ibid. V. 289, 1829); and also shewed that the envelope of a line cut *harmonically* by four fixed lines is a conic touching them (p. 294). The property of which this last is a special case was at length completely stated, simultaneously with its reciprocal (*a*), by Steiner and Chasles. It might have been deduced at once by projection from Lambert's solution of Sir Christopher Wren's problem [p. 296].

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* *Propriétés projectives* p. 115 (1822); vol. I. p. 111 (1865).

† This is obvious from Newton's figure p. [lxvii] for the case of two pairs of *parallel* tangents: it then follows by projection for any two pairs of tangents, the "cross-ratio" having *different* but constant values for different planes. See Art. 133 (ii) [p. 312].

‡ Apollonii Pergæi *De Sectione Rationis &c.* lib. I. pp. 64—5, ed. Halley (Oxon. 1706).

(*c*). *A conic regarded as the projection of a circle.*

Chasles, in his *Sections Coniques* (at the suggestion of M. J. Delbalat), defines a conic as the projection of a circle (p. 7), deduces its anharmonic properties, and founds his treatise upon them. The effectiveness of this method and the ability with which he applies it are known to all. Nevertheless, however excellent in a supplementary course of geometry, it is less suited for beginners, owing to the difficulty of proving conversely that every conic—secondarily defined by the anharmonic properties—can be placed in perspective with a circle. The problem is indeed solved concisely on p. 5, but not without references to a later paragraph and a separate work for further reasoning in justification of the construction. It naturally presents some difficulty to the tyro, being in fact a form of a problem which no geometer was able to solve generally before Desargues.

(*d*) *Homographic figures in two and in three dimensions.*

The general principle of "homographie"—as it was named by Chasles—is somewhat obscurely set forth in the works of Desargues, who regarded figures in homology as special cases of figures in perspective in space, at the same time taking for his *Proposition Fondamentale de la pratique de la Perspective* a form of the property of the anharmonic pencil.* The idea of transforming solid figures also is briefly hinted at by Desargues [p. 329]. Poncelet studied the relations of figures in "homology" (to use his own expression), and devoted a supplement of his *Traité des Propr. Projectives* pp. 369–416 (1822) to the projective properties of figures in space. Not the least valuable part of the *Aperçu historique* is the full exposition of the principle of *Homographie*, as applied to plane and to solid figures, with which it concludes.

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* Poudra *Œuvres de Desargues* I. 425.

§10. The following works may be mentioned as having advanced the knowledge of the new geometry in this country. The essay on *Transversals* in the 12th edition of Hutton's Course of Mathematics, by T. S. Davies, also an editor of the journal the *Mathematician*; Salmon's compendious works on the various geometrical and other methods, to which we have so often referred; Gaskin's *Geometrical Construction of a Conic Section;* Mulcahy's *Principles of Modern Geometry;* and Townsend's two volumes on the *Modern Geometry of the Point, Line and Circle*, which within their prescribed limits are as complete an exposition of the principles of the subject as could be desired. We must not omit to notice also Prof. H. J. S. Smith's article on the *Focal Properties of Homographic Figures* in vol. II. 196—248 of the *Proceedings* of the London Mathematical Society, some of the results of which are given below, with especial reference to the case of reverse figures.

Properties of reverse figures.

§11. If MPN be any angle which is EQUAL to its reverse mpn, it must be equal to MON [p. 323]. Hence (1) for a given position of P there are an infinity of angles MPN, each of which reverses into an equal (or supplementary) angle; and the arms of such a system of angles constitute a pencil in involution, since the points MN always lie on a circle of the coaxal system through O and P. (2) If O' and ω' be the points such that the base-line bisects OO' and $\omega\omega'$ orthogonally, every angle subtended at O or O' reverses into an equal angle subtended at ω or ω'. (3) Every conic which has O or O' for a focus reverses into a conic having ω or ω' for a focus [p. 317 (i)]. (4) An ellipse (or hyperbola) having O and O' for foci reverses into a hyperbola (or ellipse) having ω and ω' for foci; their normals at reverse points Pp correspond—being fourth harmonics to the focal distances and tangents at P and p respectively; and therefore their *centres of curvature* and their *evolutes* correspond. (5) The coaxal circles of which OO' are the limiting points reverse into those of which $\omega\omega'$ are the limiting points. (6) The parallels through ω and ω' to the base-line are such that every segment of either reverses into an equal segment. And (7) on any two reverse lines MP and mp there are two sets of equal corresponding segments, which determine two pairs of involutions having their centres on the base-line—as may readily be deduced from the constancy of the product $MP.mp$ [p. 328] for given positions of the two lines. These results apply *mutatis mutandis* to homographic plane figures in general, however placed.

Linkages.

§12. The organic description of curves has within the last few years received developments of the greatest theoretical interest and practical importance, consequent upon the discovery (1864) by Peaucellier, an officer of engineers in the French army, of an apparatus for the *Inversion* of circular into rectilinear motion. Let AOB be an angle of equal arms, and $ACBP$ a rhombus whose sides are less than the arms of the angle. Then OCP is a straight line, and the product $OC.OP$, being equal to $OA^2 - AC^2$, will be *constant* if the sides of the rhombus and of the angle be constant. Let these be now replaced by bars or "links" jointed at the five points $OABCP$, then the whole linkage is called a *Peaucellier cell.*

If this linkage be moved about a fixed pivot at O in any possible manner in one plane, then whatever be the locus of C the point P will trace its inverse with respect to O, on account of the constancy of the product $OC.OP$. To make P describe a *straight line* we must make C describe a *circle* through O; which is at once effected by joining C to an "extra link" CQ, whose end Q works about a fixed point at a distance equal to its own length from O. This apparatus may evidently be applied also to produce Parallel Motion; and we may make P describe an arc of a circle of as great a radius as we please by making the distance OQ sufficiently *nearly* equal to the length of the "extra link." The principle of linkages is well explained by Mr. A. B. Kempe in his concise work *How to draw a Straight Line, a lecture on Linkages* (London 1877), and references are given in it to the chief articles that had been written upon the subject. To conclude, in the words of Sylvester,* to whom Peaucellier's method of linkages owes so much of its further development: "*It is possible, by means of an apparatus consisting exclusively of rigid rods, compass joints, and pivots, to convert circular into linear, hyperbolic, elliptic, and parabolic motion; and, in general, to describe any curve of the form* $x\phi(x^2, y^2) + \psi(x^2, y^2) = 0$, *where* ϕ, ψ *are homogeneous forms of functions of any degree respectively in* x^2, y^2."

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* Educational Times *Reprint* vol. XXI. 58 (1874). Later information is to be sought in the same and other scientific periodicals.

NOTE.

In continuation of Note † p. lxx.

The order of the surface is thus determined by Professor Townsend. For a given position of OA, the plane OBC envelopes a quadric cone, $2n$ of whose tangent planes pass each through a generator of the director scroll. These generators give $2n$ positions of the plane ωBC and $2n$ of the point A; and when OA coincides with $O\omega$ all the A's coalesce at ω, which is therefore a $2n$-fold point on the locus. Again, every line through ω meets the scroll in n points, through each of which passes a generator; and these generators severally determine n conics, cutting the line through ω in $2n$ points, all of which, when the line is ωO, coalesce at O. Thus O also is a $2n$-fold point; and every line through O or ω passes through $2n$ other points on the locus, which is accordingly of the order $4n$.

THE GEOMETRY OF CONICS.

DEFINITIONS.

A Conic Section, or, briefly, a *Conic*, is a curve traced by a point which moves in a plane containing a fixed point and a fixed straight line in such a way that its distance from the fixed point is in a constant ratio to its perpendicular distance from the fixed straight line. The Conic Sections were so named from the circumstance that they are, and were originally defined as, the plane sections of a cone.

The fixed point is called the *Focus*; the fixed straight line the *Directrix*; and the constant ratio the *Eccentricity*, or the *Determining Ratio*.

A Conic is called an *Ellipse*, a *Parabola*, or a *Hyperbola*, according as its eccentricity is less than, equal to, or greater than unity.

Similar Conics are such as have the same eccentricity.

The *Axis* is the straight line through the focus at right angles to the directrix, and the point between the focus and the directrix in which it cuts the conic is called the *Vertex*.

When the eccentricity is either greater or less than unity, the conic cuts its axis in a second point, which is also called a vertex. In such cases the term *Axis* may denote the *finite* straight line which joins the vertices. Its middle point is called the *Centre* of the conic, and the conic is called a *Central Conic*.

The *Latus Rectum*, or, as it is sometimes called, the *Parameter*, is the chord through the focus at right angles to the axis

Other uses of these terms will be noticed in the course of the work.

A *Diameter* is the locus of the middle points of a system of parallel chords. It will be shewn that the diameters of conics are straight lines. The points in which diameters and chords meet the curve are called their ends or *Extremities*. The extremities of diameters which do not meet the curve will be defined in the chapter on Central Conics. The diameter at right angles to the axis of a central conic is called the *Minor*, or *Conjugate*, *Axis*.

Two diameters are said to be *Conjugate* when each bisects the chords parallel to the other; and two chords are said to be conjugate when they are parallel to conjugate diameters.

Supplemental Chords are such as join the extremities of a diameter to a point on the curve.

A *Tangent* to a conic is the limiting position of a secant, whose two points of intersection with the curve have become coincident. Thus, if P, Q be adjacent points on the curve, and if the chord joining them be turned about P until its further extremity Q coincides with P, the chord in its limiting position will have become the tangent at P. Hence a tangent is said to be a straight line which passes through two *consecutive* or *coincident* points on the curve.

If the point of contact of a tangent to a hyperbola be removed to infinity the tangent will coalesce with one of two straight lines through the centre, which are called *Asymptotes*.

The *Normal* at any point of the curve is the straight line drawn through that point at right angles to the tangent.

The perpendicular upon the axis from any point is called absolutely the *Ordinate* of that point; but the ordinates of a specified diameter are the segments of the chords which that diameter bisects. The term *Abscissa* will be defined later.

The portion of the axis intercepted between the tangent at any point of the curve and the ordinate of that point is called the *Subtangent*.

The portion of the axis intercepted between the normal at any point of the curve and the ordinate of that point is called the *Subnormal*.

A straight line is said to be divided *harmonically* at four points P, S, Q, R, when PQ, and PQ produced, are cut in the same ratio by S and R, so that

$$RP : RQ = SP : SQ = RP - RS : RS - RQ,$$

since, when this is the case, the lengths RP, RS, RQ are in harmonical progression, the extremes being to one another as their differences from the mean. In the case in which S is the middle point of PQ, the point R is at infinity, or $PSQ\,\infty$ is divided harmonically.

The locus of intersection of the tangents at the extremities of a chord which passes through a fixed point, or *Pole*, is called the *Polar* of the point. It will be shewn that the polar of any point with respect to a conic is a straight line; and that the polar of an external point coincides with the chord of contact of the tangents to the conic from that point.

If about any point in the plane of a conic, other than the centre of the conic, a circle be described, such that the ratio of its radius to the perpendicular distance of its centre from the directrix is equal to the *Eccentricity*, the circle may be called the *Eccentric Circle of the Conic with respect to that point*, or, briefly, the *Eccentric Circle** *of the point*. It is evident that the circle will cut, touch, or fall short of the directrix, according as the conic is a hyperbola, a parabola, or an ellipse.

The circle which is described according to the same law of magnitude about the centre of an ellipse or hyperbola is called the *Auxiliary Circle* of the curve. This latter is commonly defined as the circle described upon the axis as diameter, but it will be seen that the two definitions are coincident. The circle described upon the *Minor Axis* as diameter is called the *Minor Auxiliary Circle*.

In a central conic, the locus of intersection of tangents at right angles to one another is a circle, which is called the

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* The properties of this circle form the groundwork of the treatise of Boscovich, *Sectionum Conicarum Elementa nova quadam methodo concinnata*, contained in his *Elementa Universæ Matheseos*, Venetiis, 1757. Boscovich gave no name to his circle, but some later writers have called it the *Generating Circle*, since it affords a ready means of tracing a conic whose elements are given.

Director Circle. The corresponding locus in the parabola is the directrix.

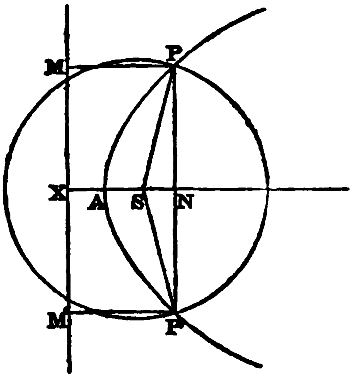
The *Order*, or *Degree*, of a curve is determined by the number of points in which it can be met by a straight line; and the *Class* of a curve by the number of tangents which can be drawn to it from a point. Thus, a curve of the second order, or degree, is one which a straight line meets generally in two, and never in more than two, points; and a curve of the second class is one to which generally two, and never more than two, tangents can be drawn from a point.

CHAPTER I.

DESCRIPTION OF THE CURVE.

1. *To trace a conic whose focus, directrix, and eccentricity are given.*

Let S be the focus,* MM' the directrix, and X the point in which the axis meets the directrix. In SX take a point A such that the ratio of SA to AX may be equal to the eccentricity. Then A is the vertex.



Let a straight line cut the axis at right angles in N. About S as centre, with radius SP, such that

$$SP : NX = SA : AX,$$

describe a circle cutting the straight line in P, P'. From these points draw perpendiculars PM, $P'M'$ to the directrix. Then evidently

$$SP : PM = SA : AX,$$

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* The planets describe approximately ellipses about the sun in one focus. For this reason the first letter of Sol is here used, as by Newton, to denote the *Focus*, or as he called it, the *Umbilicus*.

or P is a point on the curve. And in like manner P' is a point on the curve.

If now we suppose the straight line PNP' to move parallel to itself, the points P, P' upon it will trace out the entire curve. From this construction it is evident that the curve is symmetrical with respect to its axis, since its points are always determined in pairs, as P, P', which are symmetrically situated with respect to the axis. It appears also that the tangent at the vertex is at right angles to the axis, since when the point N coincides with the vertex, $SP = SA = SP'$; that is to say, the points P, P' coalesce at A, and the chord joining them, which is always at right angles to the axis, becomes the tangent at A.

In order that pairs of real points may be determined by the above construction, it is necessary and sufficient that SN should be less than SP, and therefore

$$SN : NX < SA : AX,$$

a condition which enables us to discriminate between the three species of conics as follows:

(i) *The Parabola.*

If the eccentricity be equal to unity, we must have $SN < NX$, a condition which is satisfied by taking N anywhere in XA produced. The point N therefore may be supposed to start from A, and to move in the direction AS to infinity, so that the extremities of the chord PP' trace out a single infinite branch.

(ii) *The Ellipse.*

If the eccentricity be less than unity, the curve will have a second vertex A' in XA produced, and in order that the condition

$$SN : NX < SA : AX$$

may be satisfied, it may be shewn that the point N must be taken between A, A'. Hence the ellipse consists of one oval branch, as in the figure of Art. 3.

(iii) *The Hyperbola.*

If the eccentricity be greater than unity, the curve will have a second vertex A' lying in AX produced beyond the directrix; and the point N may lie anywhere in AA' produced either way, but not between A, A'. Hence, the hyperbola consists of two infinite branches situated on opposite sides of the directrix.

A point is said to lie WITHIN a conic when it lies between the extremities of a chord perpendicular to the axis; and all other points in the plane of the conic, with the exception of those which are upon the curve itself, are said to lie WITHOUT the conic.

Let ON be the ordinate of an internal point O, and let NO be produced to meet the conic in P, then evidently

$$\begin{aligned} SO : NX &< SP : NX, \\ &< SA : AX. \end{aligned}$$

Next let O be an external point. Then if N, the foot of its ordinate, fall within the curve, it may be shewn in like manner that

$$SO : NX > SA : AX.$$

But if N fall without the curve, then

$$SN : NX > SA : AX,$$

and *a fortiori* $\quad SO : NX > SA : AX.$

Hence, in every case, *a point will lie within or without a conic according as the ratio of its focal distance to its perpendicular distance from the directrix is less or greater than the eccentricity.*

SCHOLIUM.

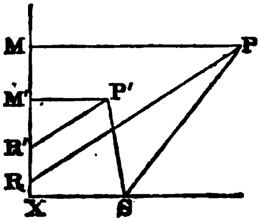
THE CIRCLE is the limiting form of an ellipse whose eccentricity is indefinitely diminished, and whose directrix is removed to an infinite distance from the focus. For if, in the next figure, PM, $P'M'$ be perpendiculars on the directrix from any two points P, P' on a conic, and if the distance of the directrix from S be increased indefinitely whilst SP, SP' remain finite, then (i) the ratio $SP : PM$ is diminished indefinitely; and (ii) the ratio $PM : P'M'$ tends to equality. But

$$SP : SP' = PM : P'M'.$$

Therefore ultimately $SP : SP'$ is a ratio of equality, and the conic becomes a circle about S as centre.

2. *The focal distances of all points on a conic are to one another as their parallel distances from the directrix.*

Let S be the focus; P, P' any two points on the curve;



M, M' their projections upon the directrix. Then from the definition

$$SP : PM = SP' : P'M'.$$

From P, P' draw a pair of parallels meeting the directrix in R, R'. Then by similar triangles,

$$PM : PR = P'M' : P'R'.$$

Therefore $$SP : PR = SP' : P'R'.$$

Hence the focal radii SP, SP' are to one another as the parallels PR, $P'R'$; and, whatever be the position of the point P on the curve, the ratio of SP to PR will be constant if PR be drawn to meet the directrix at a constant angle.

3. *A conic is a curve of the second order.*

For if P, Q be any two points on a conic, as in the figure of Art. 4, and if the straight line joining them meet the directrix in R, then, drawing perpendiculars PM, QN to the directrix, we have

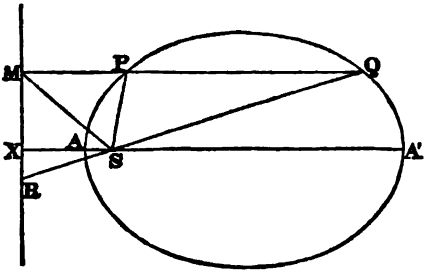
$$\begin{aligned} SP : SQ &= PM : QN \\ &= PR : QR. \end{aligned}$$

Hence SR makes equal angles with SP, SQ; and, conversely, if P be a point on the curve, and SQ be drawn meeting RP, and equally inclined with SP to SR, then Q will be a point on the curve.

It is evident from this construction that no third point can be found on the conic in the same straight line with P, Q.

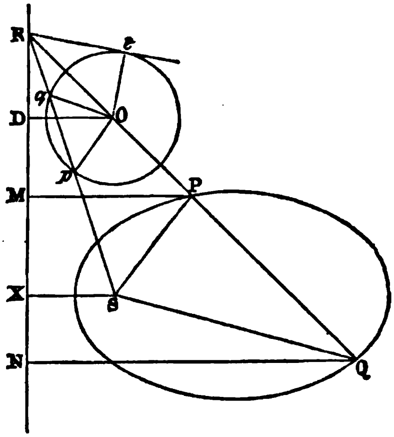
Hence a straight line which meets a conic will in general meet it in two points, and no straight line can meet a conic in more points than two. It is for this reason that conics are called curves of the second order, or of the second degree.

Let a straight line *parallel to the axis* meet the directrix in M and the curve in P. Make the angle MSR equal to MSP, and let RS meet MP in Q; then, from above, the point Q lies on the curve. In the case of the *ellipse* the points P, Q will lie on the same side of the directrix; for, since SP is less than PM, the angle SMP is less than MSP, and therefore the alternate angle MSX is less than MSP or MSR. Hence the straight line SR falls without the angle MSX, and meets MP on the same side of the directrix with P. By similar reasoning it may be shewn that a straight line parallel to the axis of a *hyperbola* intersects the curve in two points on opposite sides of the directrix. In the case of the *parabola*, SR coincides with the axis, to which MP is parallel. Hence a straight line parallel to the axis of a parabola meets the curve in one point only.



4. *To describe a conic of given focus, directrix, and eccentricity by means of the eccentric circle of any given point.*

Describe the eccentric circle of any point O in the plane of the conic, and let a straight line through S meet the circle in p and the directrix in R. Let RO meet the focal radius parallel to pO in P, and let OD, PM be the perpendiculars from O, P to the directrix.



Then by parallels,

$$SP : Op = PR : OR$$
$$= PM : OD,$$

or
$$SP : PM = Op : OD$$
$$= \text{the eccentricity}.$$

Hence, as p moves round the circle, P traces the conic which was to be described.

In the case of the hyperbola* it may be seen that the directrix divides the circle into two parts, each of which corresponds to one branch of the curve.

5. *To determine the points in which a given straight line intersects a conic of given focus, directrix, and eccentricity.*

Let the given straight line meet the directrix in R. Describe the eccentric circle of any point O on the straight line, and let it cut SR in p, q. Let the focal radii parallel to pO, qO

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* Since the locus of p is a continuous curve, the conic, which is the locus of P, is also to be regarded as in all cases a continuous curve. In the case of the hyperbola, as the point p crosses the directrix, the point P passes from infinity on one side of the axis to infinity on the other side of the axis. Hence the two branches of the hyperbola may be conceived of as connected *diagonally* at infinity.

meet the straight line in P, Q. Then, as above, if OD, PM be perpendiculars on the directrix,

$$\begin{aligned} SP : PM &= Op : OD \\ &= \text{the eccentricity,} \end{aligned}$$

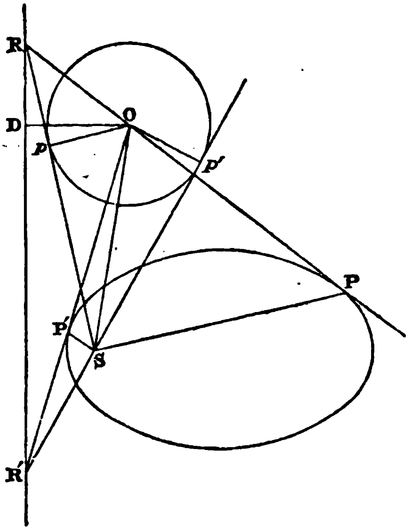
or P is a point on the conic.

Similarly it may be shewn that Q is a point on the conic.

From this construction it follows that a conic is a curve of the same order as the circle; that is to say, it is a curve of the second order, as was shewn in Art. 3.

6. *A conic is a curve of the second class.*

If the points p, q become coincident, the points P, Q likewise become coincident, since Op, Oq are always parallel to SP, SQ respectively; that is to say, if SR touches the circle, RO touches the conic.



Hence the problem of drawing tangents to a conic from a point O is reduced to that of drawing tangents from S to the eccentric circle of O; for if the tangents from S to the circle meet the directrix in R, R', then RO, $R'O$ will be the required tangents to the conic.

Since the same number of tangents can be drawn from O to the conic as from S to the circle, it follows that a conic is a curve of the same class as the circle; that is to say, it is a curve of the second class.

In order that two real tangents to the conic may be determined by the above construction, it is necessary and sufficient that S should lie without the circle. The point O must therefore be so situated that SO may be greater than the radius of the circle, and therefore

$$\begin{aligned} SO : OD &> Op : OD \\ &> \text{the eccentricity}, \end{aligned}$$

where D is the projection of O upon the directrix.

When O is on the curve the circle passes through S, and the two tangents coalesce.

No tangent can be drawn to a conic from any point between the curve and its axis, since at every such point

$$SO : OD < \text{the eccentricity}.$$

Hence no tangent can pass between the curve and its axis,* and the curve is therefore concave at all points to its axis.

EXAMPLES.

1. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse will lie wholly within the parabola, and the parabola wholly within the hyperbola; and no two conics which have the same focus and directrix can intersect one another.

2. If parallels from the focus and any point P on a conic meet the directrix in D, R, and if L be equal to half the latus rectum, then

$$SP : PR = L : SD.$$

---

* In particular it is to be noticed that no tangent can be drawn to either branch of a hyperbola from any point within the other branch.

3. If the focus and two points of a conic be given, the directrix will pass through one of two fixed points.

4. If PN be the ordinate of any point P on a conic, then

$$SP \pm L : SN = \text{the eccentricity}.$$

Hence shew that if PSP' be any focal chord, then

$$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{L}.$$

5. Determine the condition that the chord of a conic may be greater than, equal to, or less than the diameter of the eccentric circle of its middle point.

6. In the figure of Art. 4, if $OpSP$ be a quadrilateral formed by drawing through O, S a pair of parallels, and a pair of straight lines which intersect on the directrix, then p will lie without or within the eccentric circle of O, according as the ratio of SP to PM is greater or less than the eccentricity. Prove also by means of this construction that a tangent at any point to a conic cannot meet the curve in any other point.

7. If p be made to describe a series of circles about O as centre, P will describe a series of conics having a common focus and directrix; and the eccentricities of the conics will be to one another as the radii of the circles.

8. If p be made to describe a curve of any degree, P will describe a curve of the same degree; and the corresponding arcs of the two curves will subtend equal angles at the points O, S respectively.

9. If pm be the perpendicular from p to the directrix, then $PM.pm = OD.SX$. Hence shew that the sum of the reciprocals of the segments of a focal chord of a conic is constant, and any focal chord is divided harmonically by the focus and the directrix. Shew also that if $OP = OQ$, then $RqSp$ is divided harmonically.

10. Shew from the construction of Art. 6 that the tangents OP, OP' subtend equal angles, and that RP, RP' subtend right angles, at the focus.

CHAPTER II.

THE GENERAL CONIC.

In this chapter we shall prove some of the principal properties which are common to the Parabola, the Ellipse, and the Hyperbola, reserving for future consideration the properties which are distinctive of the three species of conics.

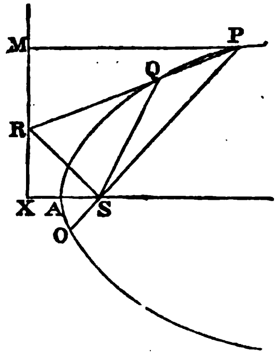
PROPERTIES OF TANGENTS.

PROPOSITION I.

7. *The tangents to a conic from any point on the directrix subtend right angles at the focus.*

Let P, Q be adjacent points on the curve, and let PQ produced meet the directrix in R. Then, as in Art. 3,

$$SP : SQ = PR : QR,$$



and SR bisects the angle which SQ makes with PS produced.

Let PS produced meet the conic in O. Then since the angles RSQ, RSO are always equal, therefore in the limit, when SQ coincides with SP, each of these angles becomes a right angle, and *RP*, which becomes the tangent at *P*, subtends a right angle at *S*.

Hence, (i) to draw the tangent to a conic at a given point *P* on the curve, make *PSR* a right angle, and draw *PR* to the point in which *SR* meets the directrix; and (ii) to draw tangents to a conic from a given point *R* on the directrix, draw the focal chord *OSP* at right angles to *SR*, and join *RP*, *RO*.

Corollary.

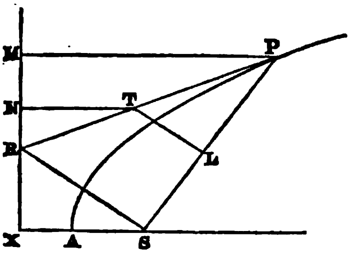
Hence it appears that the tangents at the extremities of any focal chord *PO* meet at a point *R* on the directrix; and conversely, if tangents be drawn from any point *R* on the directrix their chord of contact *PO* will pass through the focus.

The Directrix is therefore the Polar of the Focus.

PROPOSITION II.

8. *If from any point T on the tangent at P perpendiculars TL, TN be drawn to SP and the directrix respectively, then SL : TN = the eccentricity.*[*]

(i) For if the tangent at *P* meet the directrix in *R*, and



if *PM* be a perpendicular to the directrix, then, since *SR* is at right angles to *SP*, and is therefore parallel to *TL*, we have

$$SL : SP = TR : PR$$
$$= TN : PM.$$

---

[*] It will be shewn at the end of the chapter that this theorem, which, with its applications as in the text, was discovered by Prof. Adams, is the geometrical analogue of the polar equation between *ST* and its inclination to the axis.

Therefore $$SL : TN = SP : PM$$
$$= SA : AX,$$
where A is the vertex, and X the foot of the directrix.

(ii) It appears from the above proof, that this proposition may be regarded as a corollary from the preceding; but the two may be proved at once, as follows, if we consider the tangent to be defined *mutatis mutandis* after the manner of EUCLID.

Let P be a point on the curve, and R a point on the directrix, such that PR subtends at right angle at S. Take any point T in the same straight line with P, R, and let fall the perpendiculars PM, TN on the directrix, and the perpendicular TL on SP. Then, as before,
$$SL : TN = SA : AX.$$
Hence $$ST : TN > SA : AX,$$
and the point T lies without the curve in every case except that in which it coincides with P. The straight line PR is therefore the tangent at P.

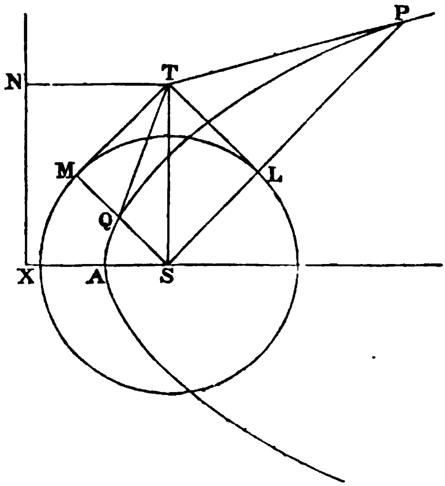
Corollary.

It is evident that if L, N be the projections of a point T upon a fixed focal chord and the directrix respectively, and if
$$SL : TN = SA : AX,$$
the point T will lie on the tangent at one or other of the extremities of the fixed focal chord.

Hence, a second construction analogous to that of Art. 6, for drawing tangents to a conic from a given point T. About S describe a circle equal to the eccentric circle of T, and draw TL, TM touching the circle at L, M; then SL, SM will pass through the points of contact of the two tangents which can be drawn to the conic from T. There is an apparent ambiguity in this construction, since each of the focal chords through L, M meets the conic in two points; but to determine the actual tangents, draw SR at right angles to SL to meet the directrix, and join RT; and draw SR' at right angles to SM to meet the directrix, and join $R'T$.

PROPOSITION III.

9. *The two tangents which can be drawn to a conic from any external point subtend equal or supplementary angles at the focus.*



For if TP, TQ be the two tangents, and TL, TM, TN be perpendiculars upon SP, SQ, and the directrix respectively, then since T lies on the tangent at P,

$$SL : TN = SA : AX.$$

In like manner

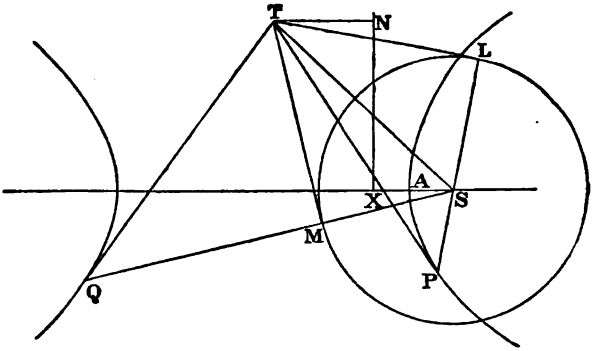
$$SM : TN = SA : AX,$$

since T lies on the tangent at Q. Therefore in the right-angled triangles STL, STM, the sides SL, SM are equal; and the hypotenuse ST is common to the two triangles; therefore

$$\angle TSL = TSM.$$

Now (i) if TP, TQ touch the same branch of the conic, the angles which they subtend at S will be either *equal* to TSL and TSM, as in the above figure, or *supplementary* to TSL and TSM. In either case TP, TQ will subtend EQUAL angles at S.

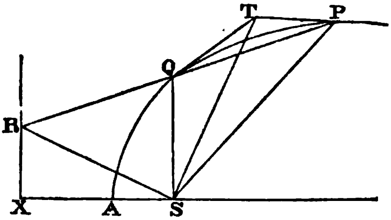
But (ii) if TP, TQ touch opposite branches of a hyperbola, so that one, and one only, of the radii SL, SM has



to be produced backwards to *P* or *Q*, then the angles *TSL*, *TSM* being equal as before, the tangents *TP*, *TQ* will subtend SUPPLEMENTARY angles at *S*.*

Corollary 1.

If the chord of contact *PQ* of a pair of tangents *TP*, *TQ* meet the directrix in *R*, then *ST*, *SR* bisect supplementary angles



at *S*, and are therefore at right angles to one another. And the chord of contact *PQ* is divided internally and externally in the same ratio *SP* : *SQ*, that is to say, it is divided harmonically, at the points at which it meets *ST* and the directrix. Since the straight line *ST* is evidently the polar of *R*, it follows that the chord *PQ* is cut harmonically by the point *R*, and the polar of *R*.

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* But in this case also, we may say that they subtend EQUAL angles, if, in accordance with the principle of the Note on Prop. VII, we regard *TQ* as subtending at *S*, not the angle *TSQ*, but its supplement.

Corollary 2.

If O be any point on PQ, or PQ produced, and M the projection of O upon the directrix, and if the perpendicular from O to ST meet SP, or SQ, in L; then, since this perpendicular is *parallel to* SR, it follows, precisely as in Prop. II., that

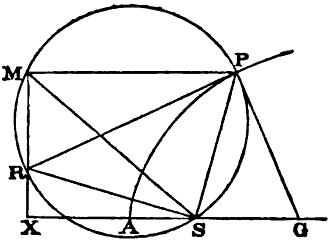
$$SL : OM = \text{the eccentricity}.$$

THE NORMAL.

PROPOSITION IV.

10. *If the normal at P meet the axis in G, then $SG : SP =$ the eccentricity.*

For if the tangent at P meet the directrix in R, the circle on PR as diameter will pass through S, since the angle PSR is a right angle; and likewise through M, the projection of P upon the directrix; and PG, which is at right angles to PR, will touch the circle.



Therefore $\angle SPG = SMP$, in the alternate segment.

Also $\angle PSG = SPM$, by parallels.

Hence the triangles SGP, PSM are similar, and

$$SG : SP = SP : PM = SA : AX.$$

Conversely, if in AS produced a point G be taken such that

$$SG : SP = SA : AX,$$

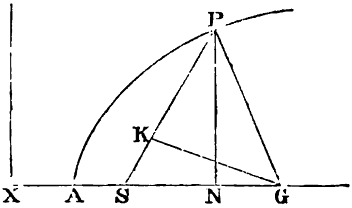
then will PG be the normal at P.

This suggests an obvious method of drawing a normal to a conic at a given point on the curve, or from a given point on the axis.

PROPOSITION V.

11. *The perpendicular let fall upon the focal radius to any point of a conic from the foot of the normal at that point meets the focal radius at a distance equal to half the latus rectum from its extremity.*

Let G be the foot of the normal at a point P whose ordinate is PN, and let a perpendicular GK be drawn to SP. Then



by similar right-angled triangles

$$SK : SG = SN : SP.$$

Therefore by the preceding proposition

$$SK : SA = SN : AX.$$

But, from the definition of the curve,

$$SP : SA = NX : AX.$$

Therefore $\quad SP \sim SK : SA = SX : AX.$

Therefore $SP \sim SK$, or PK, is constant, and equal to half the latus rectum.

ANGLE PROPERTIES OF SEGMENTS.

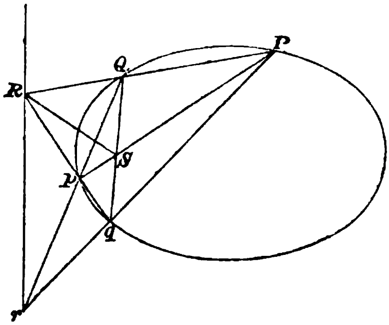
PROPOSITION VI.

12. *The chords containing the angles in a focal segment of a conic intercept on the directrix lengths which subtend right angles at the focus.*

Let PSp be a focal chord, and PQp an angle which it subtends at the circumference. Let PQ, Qp meet the directrix in R, r respectively. Produce QS to q.

Then since $\quad SP : SQ = PR : QR,$

and $\quad Sp : SQ = pr : Qr,$



therefore SR and Sr bisect the supplementary angles which Sp makes with Qq, and consequently the angle RSr is a right angle, or Rr subtends a right angle at S.

Corollary.

The opposite sides of a quadrilateral whose vertices are at the ends of a pair of focal chords PSp, QSq intersect upon the directrix, and the portion of the directrix which they intercept subtends a right angle at the focus. For, proceeding as above, we see that each of the straight lines PQ, qp meets the directrix on the bisector of the angle pSQ; and each of the straight lines Pq, Qp meets the directrix on the bisector of the supplementary angle pSq; that is to say, the two pairs of opposite sides of the quadrilateral intersect upon the directrix at points R, r, such that Rr subtends a right angle at S.

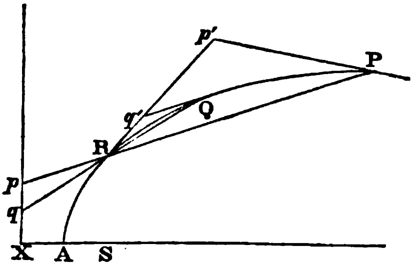
PROPOSITION VII.

13. *The chords containing the angles in a fixed segment of a conic intercept on the directrix lengths which subtend constant angles at the focus, the constant angles being equal or supplementary to half the angle which the chord of the segment subtends at the focus.*

Let PQ be a fixed arc of a conic, and PRQ a variable angle at the circumference. Let PR, QR meet the directrix in p, q respectively.

Then since $SP : SR = Pp : Rp$,

the straight line Sp bisects the angle RSP, and likewise the straight line Sq bisects the angle RSQ, externally or internally.



Hence, by addition or subtraction, as the case may be, the angle pSq is equal or supplementary* to $\frac{1}{2}PSQ$. For example, in the figure drawn,

$$\angle RSq = \tfrac{1}{2} \text{ supplement of } RSQ,$$

and

$$\angle RSp = \tfrac{1}{2} \text{ supplement of } RSP;$$

whence, by subtraction,

$$\angle pSq = \tfrac{1}{2}PSQ.$$

Corollary.

In like manner it may be shewn, by successive applications of Prop. III., that if the tangents at P, Q meet the tangent at R in p' and q', the angle $p'Sq'$ will be equal or supplementary to pSq, or $\frac{1}{2}PSQ$.

SCHOLIUM A.

The Angle Properties of conics comprise some simple generalisations of fundamental theorems in the geometry of the circle, as may be seen by removing the directrix to infinity, when, as has been already shewn, the conic becomes a circle about S as centre.

(i) Removing the directrix to infinity, we have, referring to the figure of Art. 8, the tangent PR parallel to SR, and therefore at right angles to SP. That is to say, at any point P on a circle the tangent is at right angles to the radius.

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* The theorem appears to fail when P, Q are on opposite branches of a hyperbola, in which case $\angle pSq$ = complement of $\frac{1}{2}PSQ$. But in this case the chord of the segment is not the finite straight line PQ, which lies without the conic, but the portion of the unlimited straight line through P, Q which falls within the conic. The angle subtended at S by the chord of the segment is therefore not PSQ, but the supplement of PSQ.

(ii) Removing the directrix to infinity in Art. 12, we have PQ parallel to SR, since R is at infinity; and Qp parallel to Sr, since r is at infinity. Therefore PQ, Qp contain an angle equal to RSr, or the angle in a semicircle is a right angle.

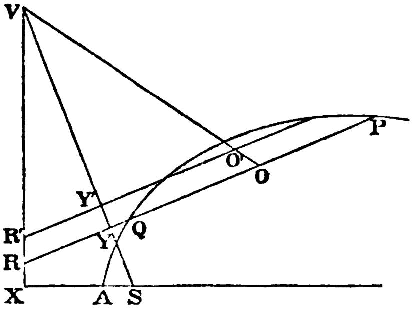
(iii) Proceeding similarly with reference to Art. 13, we have PR parallel to Sp, since p is at infinity; and QR parallel to Sq, since q is at infinity. Therefore the angle PRQ, or its supplement, is equal to $\frac{1}{2}PSQ$. Hence, by varying the positions of the points upon the circumference, we come to the properties of the circle, that the angle at the centre is double of the angle subtended by the same arc at the circumference; that angles in the same segment are equal to one another; and that the opposite angles of an inscribed quadrilateral are together equal to two right angles. Lastly, by making R coalesce with P, we deduce that the tangent at P makes with a chord PQ an angle equal to PRQ in the alternate segment.

DIAMETERS.

PROPOSITION VIII.

14. *The locus of the middle points of any system of parallel chords of a conic is a straight line which meets the directrix on the straight line through the focus at right angles to the chords.*

Let PQ be any one of a system of parallel chords, and V the point in which the focal perpendicular upon them meets the directrix. Let PQ meet SV in Y, and the directrix in R.



Then since $SP : PR = SQ : QR$;

therefore $SP^2 \sim SQ^2 : PR^2 \sim QR^2 = SP^2 : PR^2$,

or, subtracting SY^2 from each of the magnitudes SP^2 and SQ^2,

$$PY^2 \sim QY^2 : PR^2 \sim QR^2 = SP^2 : PR^2.$$

But, if O be the middle point of PQ, the sum of PY and QY will be equal to $2OY$, and their difference to PQ, or *vice versa.*

Therefore $$PY^2 \sim QY^2 = 2OY.PQ,$$

and in like manner $$PR^2 \sim QR^2 = 2OR.PQ.$$

Therefore $$OY : OR = SP^2 : PR^2,$$

which, by Art. 2, is a constant ratio for all parallel chords.

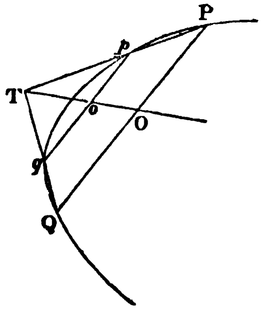
Hence the locus of O is a straight line through V.*

Corollary 1.

The tangents at the extremities of diameters are parallel to the ordinates of those diameters, since a bisected chord as PQ may be supposed to move parallel to itself until its segments vanish together, and its extremities coalesce, viz. at the end of its diameter. Hence the diameter through the point of contact of any tangent meets the directrix at a point V such that SV is perpendicular to the tangent. If a diameter meets the curve in two points, the tangents at those points are parallel to one another, and to the ordinates of that diameter. Conversely, the chord of contact of any two parallel tangents is a diameter.

Corollary 2.

If POQ, poq be double ordinates of a given diameter Oo, then since PQ, pq are both bisected by the same diameter,



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* This may also be proved by means of the eccentric circle of O. For in Art. 16, if OP be made equal to OQ, then $Sp : Sq = pR : qR$, or $RpSq$ is divided harmonically;

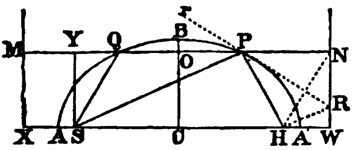
the directions of Pp, Qq will intersect at some point T on that diameter. Hence, making pq coalesce with PQ, so that PT, QT become the tangents at P, Q, we see that *the tangents at the extremities of any chord meet upon the diameter which bisects the chord;* and conversely, that the diameter through an external point bisects the chord of contact of the tangents from that point.

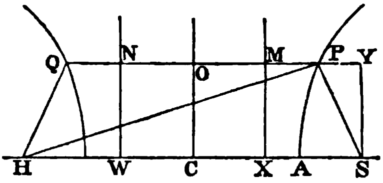
Corollary 3.

If the chord PQ be parallel to the axis, so that SY the focal perpendicular upon it is parallel to the directrix, then, proceeding as before, and supposing PQ to meet the directrix in M, we have

$$OY : OM = SP^2 : PM^2;$$

and, the ratio of OY to OM being thus constant, the locus of





O is a straight line perpendicular to the axis. Let it meet the axis in C, which (Def. p. 1) is the *Centre* of the conic. Then, evidently, CO divides the curve symmetrically, since it bisects every chord PQ to which it is at right angles; and the conic has therefore a second focus H, and directrix NW, which are the exact counterparts of the original focus and directrix with reference to which the curve was considered to be described.

From the symmetry of the curve, it is manifest that every chord through the centre is bisected at that point, and hence that all diameters pass through the centre.* Other immediate

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and therefore the focal perpendicular SY is the polar of R with respect to the circle, and $OY.OR$, being equal to the square of the radius, is in a constant ratio to OR^2, if the inclination of PQ to the directrix be invariable. Therefore $OY : OR$ is a constant ratio, and the locus of O is a straight line through V.

* A diameter is somctimes defined as a straight line through the centre.

consequences of the twofold symmetry of Bifocal Conics will be assumed as self-evident in the course of the work.

In the case of the parabola, since $SP^2 : PM^2$ is a ratio of equality, $OY : OM$ and $CS : CX$ are likewise ratios of equality. Hence the parabola may be regarded as *a conic whose centre is at infinity.* Its diameters are straight lines parallel to the axis, since they all co-intersect at the infinitely distant point C on the axis; and conversely, every straight line parallel to the axis is a diameter.

Corollary 4.

In a central conic, if one diameter bisect chords parallel to a second, the second will bisect chords parallel to the former. For if the two diameters meet the directrix in V, V', and if SV be perpendicular to CV'; then, CS being perpendicular to VV', the focus is the orthocentre of the triangle CVV', or SV' is perpendicular to CV. That is to say, if CV bisects chords parallel to CV', then CV' bisects chords parallel to CV.

If CV, CV' be thus related, it is easily seen that

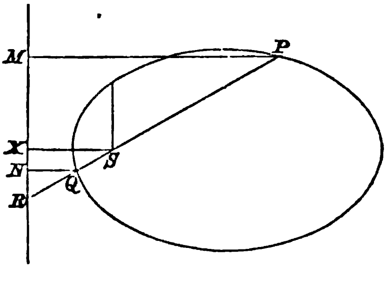
$$VX.V'X = CX.SX.$$

THE SEGMENTS OF CHORDS.

PROPOSITION IX.

15. *The semi-latus rectum is a harmonic mean between the segments of any focal chord.*

Let a focal chord PSQ meet the directrix in R, and let PM, SX, QN be perpendiculars to the directrix.



Then $$SP : SQ = PM : QN$$
$$= PR : QR,$$
or PQ is divided harmonically at S and R.

But by parallels, and from the definition of the curve, if L be the semi-latus rectum,
$$PR : SR : QR = PM : SX : QN$$
$$= SP : L : SQ.$$

And, from above, PR, SR, QR are in harmonical progression. Therefore also SP, L, SQ are in harmonical progression.

Corollary.

This result may also be written in the forms
$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{L},$$
and $$L.PQ = L(SP + SQ) = 2SP.SQ.$$

Hence, if PQ, pq be any two focal chords,
$$PQ : pq = SP.SQ : Sp.Sq,$$
or *focal chords are to one another as the rectangles contained by their segments.*

PROPOSITION X.

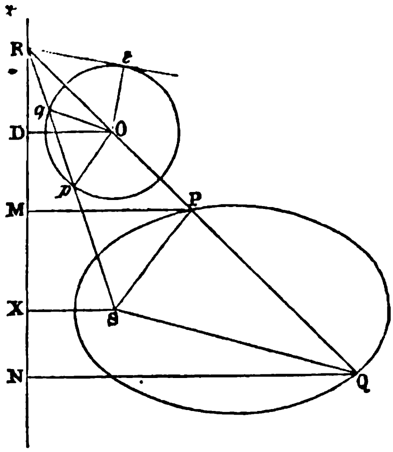
16. *A chord of a conic being divided at any point, to determine the magnitude of the rectangle contained by its segments.*

Let O be any point on a chord PQ of a conic, or on the chord produced; it is required to determine the magnitude of the rectangle $OP.OQ$.

Let the chord, produced if necessary, meet the directrix in R, and let OD be a perpendicular to the directrix. Describe the eccentric circle of O, and let it cut SR in p and q. Then, as in Art. 5, the radii Op, Oq are parallel to PS, QS respectively.

Therefore $$OP : Sp = OR : Rp,$$
and $$OQ : Sq = OR : Rq.$$

Hence $$OP.OQ : Sp.Sq = OR^2 : Rp.Rq,$$
$$= OR^2 : Rt^2,$$



if Rt be a tangent from R to the circle; or (Euclid III., 35) if it be a semi-chord at right angles to the diameter through R, in the case in which R falls within the circle.*

In this result it is to be noticed (i) that the magnitude $Sp.Sq$ depends only upon the *position* of O, since when O is given, its eccentric circle being given, $Sp.Sq$ is constant; and (ii) that the ratio $OR^2 : Rt^2$ depends only upon the *direction*† of PQ, since when the angle ORD is given, OR^2 varies as OD^2, and therefore as Ot^2, and therefore as $OR^2 \sim Ot^2$, or Rt^2.

Corollary 1.

If through any other point O' there be drawn a chord $P'Q'$ *parallel to* PQ, and if p', q' be the points corresponding to p, q, viz. on the eccentric circle of O', then, the ratio $OR^2 : Rt^2$ being the same for any two parallel chords, it follows that,

$$OP.OQ : Sp.Sq = O'P'.O'Q' : Sp'.Sq',$$

where the consequents depend only upon the positions of O, O'. If therefore any second pair of parallel chords be drawn through the same points O, O', we have the general theorem that:

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* This happens when P, Q are on opposite branches of a hyperbola, since p, q then lie on opposite sides of the directrix.

† This follows most readily in the case of the parabola, since then the circle touches the directrix in D, and the ratio in question becomes that of RO^2 to RD^2, which is constant for a given inclination of the chord.

The ratio of the rectangles contained by the segments of any two intersecting chords of a conic is equal to that of the rectangles contained by the segments of any other two chords parallel to the former, each to each.

Taking special cases, we see that this ratio is equal to that of the parallel focal chords (Prop. IX., Cor.); and to that of the squares of any pair of tangents parallel to the chords; and, in a central conic, to the ratio of the squares of the semi-diameters parallel to the chords.

Hence also, any two intersecting tangents are to one another in the subduplicate ratio of the parallel focal chords; and, in a central conic, they are in the ratio of the semi-diameters to which they are parallel.

Lastly, to take a case which will be made use of in Prop. XII., if OTO' touch a conic in T, and if OPQ, $O'P'Q'$ be a pair of parallel chords, then

$$OT^2 : O'T^2 = OP.OQ : O'P'.O'Q'.$$

Corollary 2.

If a circle and a conic intersect in four points, their common chords will be equally inclined, two and two, to the axis of the conic.* For if POQ, pOq be one of the three pairs of common chords of a circle and a conic, the rectangles $PO.OQ$ and $pO.Oq$ will be as the focal chords parallel to PQ, pq; and the same rectangles will be equal to one another, by a property of the circle. Therefore the focal chords will be equal, and therefore equally inclined to the axis.

Corollary 3.

Let the conic be a parabola,† so that the eccentric circle touches the directrix in D; and let SD meet the circle again in Z. Then, for a given inclination of the chord, the rectangle $OP.OQ$ varies as $SD.SZ$. Let V be the extremity of the diameter through D. Then, since OZ is equal to OD and parallel to VS, it is easily seen that

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* That is to say, each pair of chords will form an isosceles triangle with the axis; but they will not be parallel to one another, except when they are parallel or perpendicular to the axis.

† Another proof will be given in the chapter on the Parabola.

$$OV : SZ = VD : SD = SD : 2SX.$$

Hence $OP.OQ$ varies as $2SX.OV$, and is equal to $F.OV$, where F is the focal chord parallel to PQ.

This may also be deduced as a special case from Cor. 1, by regarding any two diameters as chords $V\infty$ and $V'\infty$, whose further extremities are at infinity; for, if the parallel chords PQ, $P'Q'$ meet the two diameters in O and O', then

$$OP.OQ : O'P'.O'Q' = OV.O\infty : O'V'.O'\infty = OV : O'V',$$

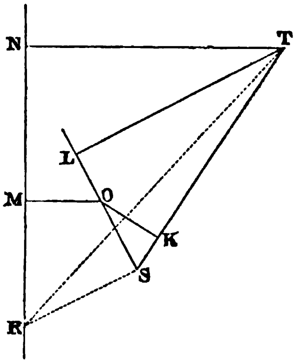
since it may be shewn that $O\infty : O'\infty$ is a ratio of equality; and therefore $OP.OQ$ varies as OV.

POLAR PROPERTIES.*

PROPOSITION XI.

17. *If a chord of a conic pass through a fixed point, the tangents at its extremities will intersect on a fixed straight line; and conversely, if pairs of tangents be drawn to a conic from points on a fixed straight line, their chords of contact will pass through a fixed point.*

If O be any point on the chord of contact of the tangents from T to a conic, and if TL be a perpendicular to SO, and



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* The theory of *Polars*, although the name is of later origin, was known to Desargues. See Poudra's *Œuvres de Desargues*, vol. I., p. 263, (PARIS, 1864).

OM, TN be perpendiculars to the directrix, then will the rectangle $SO.SL$ be in a constant ratio to $OM.TN$.

Draw SP to one of the points in which the chord meets the conic, and let TL' be a perpendicular to SP.* Upon ST let fall the perpendicular OK, and produce it to meet SP in L''. Then, since each of the ratios $SL' : TN$, and $SL'' : OM$ (Prop. III., Cor. 2), is equal to the eccentricity, the rectangle $SL'.SL''$ is in a constant ratio to $OM.TN$.

And because the angles at K, L, L' are right angles, the points K, T, L, O, and the points K, T, L', L'', are concyclic. Therefore

$$SO.SL = SK.ST = SL'.SL'',$$

which has been shewn to vary as $OM.TN$. Hence, *if O be a fixed point*, SL varies as TN, and the locus of T becomes a straight line,† which meets the directrix at a point R, such that OSR is A RIGHT ANGLE. Conversely, if T be taken on the fixed straight line TR, the chords of contact will co-intersect at O.

When the POLE O lies without the conic, its POLAR, the locus of T, is *the chord of contact of the tangents from O*, since these points of contact are evidently points on the locus.

Corollary 1.

From the above investigation it is evident that, if a point T lies on the polar of O, then O lies on the polar of T. Take any two straight lines A, B, and let a, b denote their poles. Then the polar of any point on A passes through a, and the polar of any point on B passes through b, and therefore the polar of the intersection of A, B passes through both a and b. That is to say, *the intersection of any two straight lines is the Pole of the straight line which joins their two Poles.*

Corollary 2.

Since every point at infinity in the plane of a central conic is the point of intersection of a pair of tangents whose chord of contact, being a diameter (Prop. VIII, Cor. 1), passes through the centre, all such points at infinity are on the polar of the centre, and may therefore be regarded as lying on a straight line, which is called the *Straight Line at Infinity.*

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* See the lithographed figure, No. 1.

† See Scholium B.

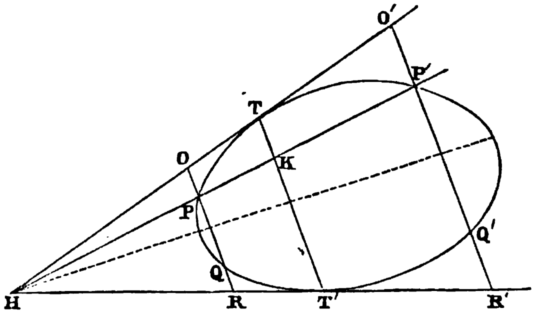
Corollary 3.

Since when O is a fixed point SL varies as TN, the straight line which is the locus of T is a tangent (Art. 8), viz. at the point in which it meets SO, to a conic having the same focus and directrix, and whose determining ratio is that of SL to TN; and further, it will be a tangent to the same conic if O be no longer fixed, but subject only to the condition that the ratio of SO to OM is constant. Hence, if a point O lie on a conic, the envelope of its polar with respect to a conic having the same focus and directrix will be a third conic having the same focus and directrix, and conversely; and the eccentricities of the three conics will be proportionals.

PROPOSITION XII.

18. *All chords drawn through any point to a conic are cut harmonically by that point, and its polar with respect to the conic.*

Let HT, HT' be a pair of tangents to a conic, and PP' a chord which passes through H, and cuts the chord of contact TT' in K; so that H is on the polar of K, and K on the polar of H. Through P, P' draw parallels to TT', and let them meet the curve in Q, Q', and the two tangents in O, O' and R, R' respectively.



Then since the straight line which bisects TT' and passes through H bisects also OR and $O'R'$; and since, by Prop. VIII., Cor. 2, the same straight line is the diameter which bisects the chords PQ, $P'Q'$; therefore the intercepts OQ, PR are equal, and likewise the intercepts $O'Q'$, $P'R'$.

Hence, and by Prop. x., Cor. 1,

$$\begin{aligned} OT^2 : O'T^2 &= OP.OQ : O'P'.O'Q' \\ &= OP.PR : O'P'.P'R' \\ &= OH^2 \quad : O'H^2, \end{aligned}$$

by similar triangles. That is to say, $HOTO'$ is cut harmonically, and therefore $HPKP'$ is cut harmonically.

Corollary.

The diameter through H is divided harmonically at that point, and the point in which it meets the double ordinate TT'. Let it meet the latter in V, and the curve in D and D'. Then, if C be the centre of the conic, and therefore the middle point of DD', it follows from the nature of harmonic section that $CV.CH = CD^2$. But in the case of the parabola, if D and ∞ be the extremities of the diameter through H, then HV is divided harmonically at D and ∞, and therefore HV is bisected at D.

SCHOLIUM B.

In Prop. XI, having shewn that SL, the projection of ST on a fixed straight line SO, varies as the perpendicular distance of T from another fixed straight line, the directrix, we inferred that the locus of T was a straight line; and that it met the directrix at a point R such that $\angle\, OSR =$ a right angle. This is virtually proved in Art. 8, where, leaving the curve out of consideration, we may regard the eccentricity as any constant ratio. In Prop. XI. there is the same ambiguity as in Prop. II, Cor., the locus of T apparently consisting of *two* straight lines through R. This arises from the circumstance that when the *magnitude* only of the ratio $SO : OM$ is given, the point O is not completely determined, but the choice lies between two points O, O' collinear with the focus, each of which has its own polar. If, however, the actual position of O be given, as in the proposition, then taking into consideration the *sign* of the ratio $SL : TN$ as well as its magnitude, let the direction SO be regarded as positive, and that of OS negative; and let perpendiculars to the directrix from its S-side be positive, and those from the further side negative. Then, SL, TN being positive or negative together, the locus of T is seen to be the single straight line TR.

The polar of O is the straight line through R *parallel to the ordinates of the diameter through O;* for if ∞ be the pole of that diameter, and V its point of concourse with the directrix, then (Prop. III., Cor. 1) $S\infty$ is at right angles to SV, and is therefore (Prop. VIII.) parallel to the ordinates of the diameter OV. In the case of central conics this follows at once from Prop. VIII., Cor. 1.

If e denote the eccentricity, then, referring to the proof of Prop. XI., we see that

$$SO.SL = e^2.OM.TN.$$

Hence (i) if SO be less than $e.OM$, then will SL, and *a fortiori* ST, be greater than $e.TN$; but (ii) if SO be greater than $e.OM$, then will SL be less than $e.TN$, and ST, which may have any magnitude not less than SL, may be either less or greater than $e.TN$. It follows that *the polar of O will cut or not cut the conic according as O lies without or within the conic.*

SCHOLIUM C.

The Polar Equation of a conic referred to its focus and axis may be seen, from Example 4, to be of the form

$$\frac{l}{r} = 1 + e\cos\theta,$$

where r denotes SP; θ the angle ASP; and e, l denote the eccentricity and the semi-latus rectum. The corresponding equations of the Tangent, the Normal, of any Chord, and of the Polar of any point, may be deduced, as below, from geometrical theorems which we have already proved.

(i) *The Tangent.*

In Prop. II., let r, θ be the coordinates of T, and let α be the angular coordinate of the point of contact P.

Then $$SL = ST\cos TSL = r\cos(\theta - \alpha),$$

and $$e.TN = e(SX - ST\cos AST) = l - e.r\cos\theta.$$

Hence $$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha).$$

(ii) *The Normal.*

In Prop. IV., let a parallel to the axis cut SP in Z, and PG in Q. Denote $\angle ASP$ by α, and let r, θ be the coordinates Q.

Then $$ZQ = e.ZP = e(SP - SZ),$$

or $$e.SP = ZQ + e.SZ;$$

and $$\frac{ZQ}{r}=\frac{\sin(\theta-\alpha)}{\sin\alpha};\quad \frac{SZ}{r}=\frac{\sin\theta}{\sin\alpha}.$$

Hence $$\frac{e\,.\,SP\sin\alpha}{r}=\sin(\theta-\alpha)+e\sin\theta;$$

or $$\frac{l}{r}\cdot\frac{e\sin\alpha}{1+e\cos\alpha}=\sin(\theta-\alpha)+e\sin\theta,$$

since SP, α are the coordinates of a point on the curve.

(iii) *Any Chord.*

From Prop. III., Cor. 2, it is easy to deduce the equation

$$\frac{l}{r}=e\cos\theta+\sec\beta\cos(\theta-\alpha),$$

representing the chord which cuts the conic at the points whose angular coordinates are $\alpha\pm\beta$ respectively.

(iv) *The Polar of any Point.*

In Prop. XI., let r, θ be the coordinates of T, and ρ, α those of O. Then it is easily seen that

$$e\,.\,TN=l-e\,.\,r\cos\theta;\quad e\,.\,OM=l-e\,.\,\rho\cos\alpha;\quad SK\,.\,ST=\rho\cos(\theta-\alpha)\,.\,r.$$

Hence, equating $SK\,.\,ST$ to $e^2\,.\,OM\,.\,TN$, we deduce that

$$\left(\frac{l}{\rho}-e\cos\alpha\right)\left(\frac{l}{r}-e\cos\theta\right)=\cos(\theta-\alpha),$$

which is the equation of the polar of the point (ρ,α).

EXAMPLES.

11. Determine the pole of the latus rectum of a conic.

12. Every tangent is the polar of its point of contact.

13. The segments of any focal chord subtend equal angles at the point in which the directrix meets the axis.

14. If two conics have a common focus, their common chord or chords will pass through the intersection of their directrices.

15. The tangents at the ends of a focal chord meet the latus rectum at points equidistant from the focus.

16. The focal distance of any point on a conic is equal to the ordinate at that point produced to meet the tangent at an extremity of the latus rectum.

17. The directions of any two tangents to a circle are equally inclined to the diameter through their point of intersection. State this theorem in a form applicable to all conics.

18. Given the focus of a conic and a focal chord, the locus of the extremities of the latus rectum is a circle.

19. Given the focus, the length of the latus rectum, a tangent, and its point of contact, shew how to construct the conic.

20. When the focus and three points of a conic are given, shew how to construct the curve.

21. Given the focus of a conic inscribed in a triangle, determine the points of contact.

22. Given a chord of a conic and the angle which it subtends at the focus, shew that the focal radius to the pole of the chord passes through a fixed point.

23. With given focus and eccentricity construct a conic which shall pass through two given points.

24. Determine in what cases a chord of a conic will be a maximum or a minimum.

25. The portion of any tangent intercepted between the tangents at the ends of the parallel focal chord is divided at its point of contact into segments whereof each is equal to the focal distance of that point.

26. If the tangent at any point of a conic meet the directrix in D, and the latus rectum in L, then

$$SL : SD = SA : AX.$$

27. If PM, QN be the ordinates of the extremities of a focal chord PQ, and if the direction of the chord meet the directrix in R, then will RN meet MP at a distance from the axis equal to $2PM$.

28. If M be the projection upon the directrix of any point P on a conic, then will SM meet the tangent at the vertex upon the bisector of the angle SPM. If a focal chord of central

conic meet the tangents at the vertices in V, V', give a construction for determining the points in which the circle on VV' as diameter meets the conic.

29. Prove the following construction for drawing tangents to a conic from a given point T. Divide ST in t, so that

$$St : ST = AX : TN,$$

where TN is a perpendicular to the directrix; about S as centre describe a circle touching the conic, and from t draw tangents to the circle, and let them meet the tangent at the vertex in V, V'; draw TV, TV', which will be the tangents required.

30. If a chord of a conic subtend a constant angle at the focus, the locus of its pole will be a conic having the same focus and directrix. Shew also that the envelope of the chord will be another conic having the same focus and directrix, and that the eccentricities of the three conics will be proportionals.

31. The vertex of a triangle which circumscribes a conic, and whose base subtends a constant angle at the focus, lies on a conic.

32. Two sides of a triangle being given in position, if the third subtends a constant angle at a fixed point, determine its envelope.

33. If a fixed straight line intersect a series of conics which have the same focus and directrix, the envelope of the tangents to the conics at the points of section will be a conic, have the same focus, and touching both the fixed straight line and the directrix of the series of conics.

34. The focal perpendicular upon any tangent to a conic is a mean proportional to the segments into which it divides the portion of that tangent intercepted between the tangents at the extremities of any focal chord.

35. If SY be the focal perpendicular on the tangent at any point P to a conic, and X the point in which the axis meets the directrix, then

$$SY : YX = SA : AX.$$

Determine the locus of Y, and shew that it is the envelope of the circle on SP.

36. If PN be the perpendicular from any point P on a conic to the latus rectum, the straight line connecting N with the point in which the axis meets the directrix will pass through the foot of the perpendicular let fall from the focus upon the tangent at P.

37. If the diameter at a point P on a conic bisects the chord normal at Q, the diameter at Q bisects the chord normal at P.

38. In Art. 10, shew that the normal PG becomes equal to the semi-latus rectum when P coincides with the vertex of the conic.

39. The perpendicular from G on SP varies as the ordinate of P; and the foot of this perpendicular lies upon the straight line which passes through the foot of the ordinate of P, and is parallel to SM.

40. If Q be any point on the normal at P, and L and M be its projections on SP and the ordinate of P, shew that

$$QL : PM = SA : AX.$$

41. The perpendicular upon a focal chord from the intersection of the normals at its extremities meets the chord at a distance from one extremity which is equal to the focal distance of the other; the locus of the foot of this perpendicular is a conic; and the straight line drawn parallel to the axis through the intersection of the normals passes through the middle point of the chord.

42. If P be the pole of a normal chord which meets the directrix in Q, shew that the circle SPQ passes through an extremity of the chord.

43. If a circle touch a conic on opposite sides of its axis, it will intercept a constant length upon the focal chords through the points of contact. When the circle passes through the focus, determine the focal radii to the points of contact.

44. The rectangle contained by the focal perpendicular upon the tangent at any point to a conic and the portion of the normal intercepted between the curve and its axis is equal to the rectangle contained by the semi-latus rectum and the focal distance of the point.

45. If QQ' be a focal chord of a conic, and if the normal at P be at right angles to the chord and meet the axis in G, then

$$PG^2 = SQ.SQ'.$$

46. Shew also that, if a parallel to the chord be drawn through G and meet the direction of PS in U, then $PU = \frac{1}{2}QQ'$.

47. If the normal to a conic at P meet the axis in G, and if SY the focal perpendicular upon the tangent meet the directrix in V, shew that

$$PG : SY = SV : VY.$$

48. The ratio of the normals, terminated by the axis, at any two points of a conic is equal to that of the tangents at those points.

49. Given an arc of a conic, shew how to construct the curve.

50. The parallel diameters of two similar and similarly situated conics bisect the same systems of parallel chords. If the two conics be concentric ellipses or hyperbolas, or equal parabolas whose axes are coincident, shew that any chord of the exterior conic is divided into pairs of equal segments by the interior, and that any chord of the former which touches the latter is bisected at the point of contact.

51. The angle between any two chords of a conic is equal to the angle subtended at the focus by the portion of the directrix intercepted by the diameters which bisect the chords.

52. The arms of the angle which a focal chord of a conic subtends at any point on the circumference meet the directrix upon diameters through the points of contact of tangents at right angles.

53. The polar of any point with respect to a conic meets the directrix on the diameter which bisects the focal chord through that point.

54. The diameter through any point, and the polar of that point, meet the directrix and the axis respectively on a straight line parallel to the focal distance of the point. Hence shew that the foot of the ordinate of any point in the plane of a central conic is at a distance from the centre which varies inversely as the distance therefrom of the intersection of the polar of the point with the axis.

55. From the preceding example deduce a construction for drawing tangents to a conic from a given point.

56. The triangle whose angular points are the focus of a conic and the intersections of the tangent and the diameter at any point with the axis and the directrix respectively has its orthocentre at the point in which the tangent meets the directrix.

57. Given the focus and the directrix of a conic, shew that the polar of a given point with respect to it passes through a fixed point.

58. If the polar of a point O with respect to a conic intersect a conic having the same focus and directrix in P, and if SQ be drawn at right angles to SP to meet the directrix in Q, the locus of the intersection of QO and SP will be a conic.

59. Deduce from Art. 16 that the square of the ordinate at any point of a conic varies either as the distance of the foot of the ordinate from the vertex, or as the rectangle contained by the segments into which it divides the axis.

60. A focal chord of a conic and the diameter which bisects it meet any fixed straight line perpendicular to the axis at points whose ordinates contain a constant rectangle; and the square of the ordinate of the middle point of the chord varies either as the distance of the foot of the ordinate from the focus, or as the rectangle contained by its distances from the focus and the centre of the conic.

61. If a chord of a conic passes through a fixed point in the axis, determine the locus of its middle point, and in the case of a central conic, the locus of its intersection with another chord which passes through a fixed point in the axis and is parallel to the diameter which bisects the former.

62. If a tangent be drawn parallel to any chord of a conic, the portion of it terminated by the tangents at the ends of the chord is bisected at its point of contact.

63. Two tangents being applied to a line of the second order, if from any point in one of them a straight line be drawn parallel to the other, the portion of it intercepted by the chord joining the points of contact will be a mean proportional to its segments made by the curve. Examine the case in which the secant becomes a tangent.

64. In Art. 16, investigate the case in which O coincides with S, and shew that SX is then a mean proportional to the distances of P and p from the directrix.

65. Shew also that, if SZ be drawn parallel to $PQOR$ to meet the directrix, then

$$OP.OQ : Sp.Sq = SZ^2 : SZ^2 - L^2,$$

where L denotes the semi-latus rectum.

66. If a chord of a conic subtends equal angles at the extremities of another chord, it likewise subtends equal angles at the extremities of any chord parallel to the latter.

67. If ABC be a triangle whose sides touch a conic at the points a, b, c, then

$$Ab.Bc.Ca = Ac.Ba.Cb.$$

68. If any conic be drawn through four given points, and if a fixed straight line meet the conic in P, Q, and one of the pairs of straight lines joining the four points in A, B, then will the ratio of the rectangle $PA.AQ$ to the rectangle $PB.BQ$ be constant.

69. Any tangent to a conic is divided harmonically by its point of contact and the three points in which it meets any two other tangents and their chord of contact. Examine the cases in which two of these four straight lines become parallel.

70. If from any point on a conic parallels be drawn to two adjacent sides of a given inscribed quadrilateral figure, the rectangles under the segments intercepted by those adjacent and by the other two opposite sides will have a given ratio.

71. If ABC be a triangle inscribed in a conic, and if from any point O on the curve there be drawn a parallel to BA meeting BC and the tangent at A in P, Q, and a parallel to BC meeting AB, AC in P', Q'; then will $OP.OQ$ be to $OP'.OQ'$ in a constant ratio, viz. that of the focal chords parallel to BA and BC respectively.

72. If from any point on a conic pairs of perpendiculars be drawn to the opposite sides of a given inscribed quadrilateral, the rectangle contained by the one pair of perpendiculars will be in a constant ratio to the rectangle contained by the other pair.

73. The perpendicular from any point on a conic to a fixed chord is a mean proportional to the perpendiculars from that point to the tangents at the extremities of the chord.

74. If from any point on a conic straight lines be drawn at given angles to two adjacent sides of a given inscribed quadrilateral figure, the rectangle under the segments intercepted by those adjacent and by the other two opposite sides will have a given ratio.

75. Hence shew that, if from a given point M there be drawn two fixed straight lines meeting a conic in A, B and C, D; and likewise a variable straight line meeting the curve in E, E', and the straight lines AC, BD in K, L; then

$$EM^2 : E'M^2 = LE.EK : LE'.E'K;$$

and investigate the form which this relation assumes when the fixed straight lines become tangents to the conic.

76. Deduce from the preceding example that, if A, B, C, D be any four points on a conic, the three straight lines joining the intersections of AB, CD; BC, DA; and CA, BD, are cut harmonically by the curve, and that each of these points is the pole of the straight line which joins the other two.

77. Hence shew how to draw tangents to a conic from any external point with the help of the ruler only.

78. If PVP' and QVQ' be any two intersecting chords of a conic, and if the circle through Q, P, Q' meet PP' in R, then

will the ratio of VP' to VR be equal to that of the focal chords parallel to PP' and QQ'. Examine the cases in which two or all of the points P, Q, Q' coalesce.

79. If PQ be any chord of a conic, and F the parallel focal chord, and if the direction of F meet the tangent at P in T, then

$$PQ.ST = F.SP.$$

80. If there be a quadrilateral figure inscribed in a conic section, and if from one of its angular points there be drawn parallels to the sides about the opposite angle; and if from the two remaining angles there be drawn straight lines to any point in the curve to meet the parallels; the intercepted portions of the parallels, estimated from their common point, will have a given ratio, wherever in the curve the fifth point be taken.

CHAPTER III.

THE PARABOLA.

19. The parabola being a conic whose determining ratio is one of equality, some of its properties may be at once deduced by equating $SA : AX$ to unity from properties of the general conic already proved; thus SL becomes equal to TN in Art. 8, and SG equal to SP in Art. 10. The semi-latus rectum of the parabola is equal to SX; that is, to $SA+AX$, or $2SA$.

Other properties of the parabola may be derived from those of central conics by regarding it as a conic whose centre* and second focus are at infinity, and the further extremities of whose diameters† are likewise at infinity; but in the present chapter we shall give independent proofs of such properties, commencing with the original definition of the parabola.

The portion of any diameter intercepted between the curve and the ordinate of any point with respect to that diameter is called the *Abscissa* or *Absciss* of the point; and any focal chord of a parabola is called the *Parameter* of the diameter which bisects it.

CHORD PROPERTIES.‡

PROPOSITION I.

20. *The ordinate of any point on the parabola is a mean proportional to the abscissa and the latus rectum.*

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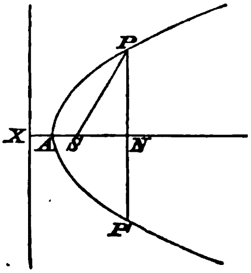
* See Art. 14, Cor. 3.

† See Art. 16, Cor. 3.

‡ Under this head are included such propositions only as can be proved antecedently to the definition of a tangent; but the restriction does not apply to the *Corollaries* from those Propositions.

Let AN be the abscissa of any point P on the curve, and X the point in which the directrix meets the axis. Then, by Euclid I. 47, and from the definition of the parabola,



$$PN^2 + SN^2 = SP^2 = NX^2.$$

Hence $$PN^2 + (AN \sim AS)^2 = (AN + AS)^2.$$

Therefore $PN^2 = 4AS.AN$, or PN is a mean proportional to AN and $4AS$, which latter, by Art. 19, is equal to the latus rectum.

Conversely, if the square of the ordinate of any point P vary as its abscissa, the locus of the point will be a parabola.

The above proposition suggests an obvious method of tracing the curve, since for any assumed magnitude of AN the magnitude of PN and the position of P are determined.

Corollary.

Hence, to find two mean proportionals between a given pair of straight lines,* with latera recta equal to the given lines describe two parabolas, having a common vertex, and their axes at right angles; then will the ordinates of either of their points of intersection be mean proportionals to their latera recta, as required; for it is evident that the ordinate in either parabola will be a mean proportional to its own latus rectum and the ordinate in the other.

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* This problem, which is of great historical interest, was solved as above by Menæchmus, according to the statement of Eutokius. Compare Bretschneider's, *Die Geometrie und die Geometer vor Euklides*, p. 160 (LEIPZIG, 1870).

PROPOSITION II.

21. *The locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to the axis; and the bisecting line meets the directrix on the straight line through the focus at right angles to the common direction of the chords.*

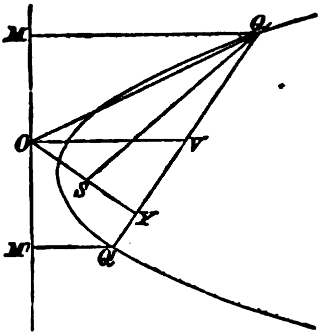
Take QQ', any one of a system of parallel chords, and let M and M' be the projections of its extremities upon the directrix.

Let the focal perpendicular upon the chords meet QQ' in Y, and the directrix in O; and through O draw a parallel to the axis meeting QQ' in V. Then will V be the middle point of QQ'.

For $$OM^2 = OQ^2 - QM^2 = OQ^2 - SQ^2$$
$$= OY^2 - SY^2;$$

and OM'^2 may be shewn to have the same value.

Therefore OM, OM' being equal, the straight line through

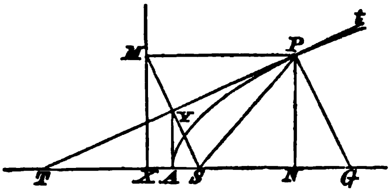


O parallel to the axis bisects QQ'; that is to say, it bisects every chord which is at right angles to OS.

Hence it is evident that every straight line parallel to the axis of a parabola is a diameter of the curve, and that all diameters are parallel to the axis and to one another.

Corollary.

It follows, as a particular case of the above proposition, that the direction of the focal perpendicular SY on the tangent at P to a parabola meets the directrix at a point M such that PM is parallel to the axis.

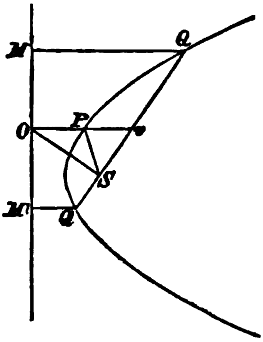


Hence it appears that the tangent at P bisects the angle SPM, as will be otherwise proved in Art. 25; and it may also be deduced, independently of Art. 7, that the intercept on the tangent made by the curve and the directrix subtends a right angle at S.

PROPOSITION III.

22. *To find the length of any focal chord of a parabola.*

Let QQ' be any focal chord; M and M' the projections of its extremities upon the directrix; and O the point in which



the focal perpendicular upon the chord meets the directrix. Let a parallel through O to the axis meet QQ' in v, which, by Prop. II, will be the middle point of the chord.

Hence, and from the definition of the curve,

$$QQ' = SQ + SQ' = QM + Q'M'$$
$$= 2vO.$$

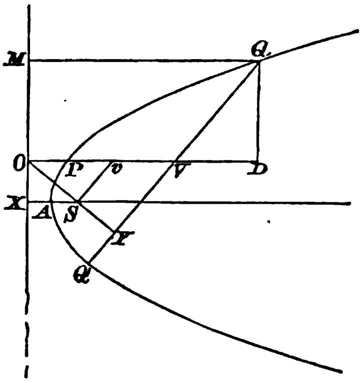
And because OSv is a right angle, and $SP = PO$; therefore $Pv = SP = PO$, and therefore $vO = 2SP$.

Hence $QQ' = 4SP$, or the parameter of any diameter of a parabola is equal to four times the focal distance of the extremity of that diameter. In particular, as we have already seen, the latus rectum is equal to $4SA$.

PROPOSITION IV.

23. *The ordinate of any point on a parabola with respect to any diameter is a mean proportional to its parameter and the abscissa of the point.*

Let QV and PV be the ordinate and abscissa of any point Q on the curve; let VP meet the directrix in O, and the focal chord parallel to QV in v; and let OS, which (Prop. II.) is at right angles to Sv and QV, meet the latter in Y.



Then, as in Art. 21, if D and M be the projections of Q on the diameter PV, and on the directrix,

$$QD^2 = OM^2 = OY^2 - SY^2.$$

And since, by similar triangles, the lengths QD, OY, SY are proportional to QV, OV, vV, therefore, from above,

$$QV^2 = OV^2 - vV^2.$$

And since $Pv = SP = PO$, as in Art. 22, the sum of OV and vV is equal to $2PV$, and their difference to $2SP$, or *vice versa*; and therefore, in either case, the difference of their squares is equal to $2SP.2PV$.

Therefore $QV^2 = 4SP.PV$, or the ordinate QV is a mean proportional to the parameter $4SP$ (Prop. III), and the abscissa PV.

Corollary 1.

It may be shewn that $QD'^2 = 4AS.PV$; and further, that if a straight line QD' be drawn *in any direction* from Q to the diameter PV, it will be a mean proportional to the parallel focal chord and the abscissa PV. This follows most readily with the help of the theorem (Art. 30, Cor. 1), that *if the base of a triangle be parallel to the axis of a parabola the squares of its remaining sides will be as the parallel focal chords.*[*]

Corollary 2.

If the tangent at P meet QM in R, then the figure $PVQR$ being a parallelogram, it follows that $RP^2 = 4SP.RQ$. Hence, if R be any point on the tangent at a given point P to a parabola, and if the diameter through R meet the curve in Q, then will RP^2 vary as RQ.

Corollary 3.

On the tangent at a given point P to a parabola take any two points T and R; and let the diameters through them meet in curve in E and Q, and let the former diameter meet PQ in F. Then, by Cor. 2, and by similar triangles,

$$TE : RQ = TP^2 : RP^2 = TF^2 : RQ^2.$$

Hence $$TE : TF = TF : RQ = PF : PQ,$$

or, *the portion of any diameter intercepted by any chord and the tangent at either extremity of the chord is divided at the curve in the same ratio as that in which it divides the chord.*

PROPOSITION V.

24. *A chord of a parabola being divided at any point, to determine the magnitude of the rectangle contained by its segments.*

Let any chord QR be divided internally or externally at the point O; and let the diameters through O and the middle

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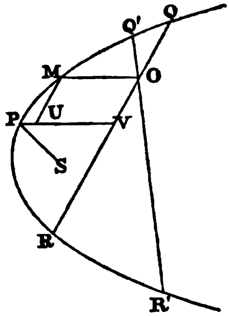
[*] This may be deduced without the help of tangent-properties from the second note on p. 28, or from Art. 22, where QQ' varies as $SQ.SQ'$, that is to say, as SO^2, the angle QOQ' being a right angle. It follows that the focal chords of a parabola vary inversely as the squares of the sines of their inclinations to the axis.



point V of the chord meet the parabola in M and P. Then by Prop. IV, and by Euclid II. 5, Cor., if MU be the ordinate of M with respect to the latter diameter,

$$\begin{aligned} QO.OR = QV^2 \sim OV^2 &= QV^2 \sim MU^2 \\ &= 4SP(PV \sim PU) \\ &= 4SP.MO, \end{aligned}$$

or the rectangle whereof the magnitude was to be found varies as MO, which depends only upon the *position* of the point O; and as the parameter $4SP$, which depends only upon the *direction* of the chord QR.

Corollary.

Hence, if $Q'R'$ be any second chord through O, and $4SP'$ the corresponding parameter,

$$QO.OR : Q'O.OR' = 4SP : 4SP',$$

or these rectangles are proportional to the focal chords parallel to QR, $Q'R'$, as was proved also for the general conic in Art. 16. Hence also, the squares of any two intersecting tangents are as the focal distances of their points of contact.

TANGENT PROPERTIES.*

PROPOSITION VI.

25. *The tangent to a parabola at any point is the bisector of the angle which the focal radius makes with the diameter produced.*

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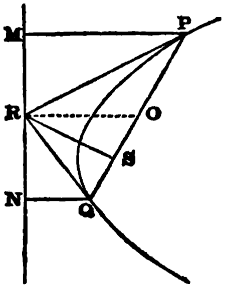
* See also the Corollaries in Articles 21, 23, 24.

(i) Let the tangent at any point *P* meet the directrix in *R*, and let the diameter produced beyond the curve meet the directrix in *M*.



Then since *PR* subtends a right angle at *S*, and since *SP*=*PM*, and *PR* is common to the right-angled triangles *SPR*, *MPR*, therefore their angles at *P* are equal, or the tangent *PR* bisects the angle *SPM*.

It is likewise evident that *SP* and *MP* make equal angles with *RP* produced towards *t*, as in the figure of Art. 21, Cor.; and that if *PR*, or *PR* produced, meet the axis in *T*, the angles at *P* and *T* in the triangle *SPT*, and therefore also the sides *SP* and *ST*, will be equal to one another.

(ii) Or we may proceed as follows, taking EUCLID'S definition of a tangent.

Draw the straight line bisecting the angle *SPM*, and take any point upon it. The distance of any such point from *S* is equal to its distance from *M*, and therefore greater than its distance from the directrix, except when the point coincides with *P*. Hence every point except *P* on the bisector of the angle *SPM* lies without the curve, and the bisector of *SPM* is therefore the tangent at *P*.

Corollary 1.

It is evident from the above that the tangent at *P* bisects the angle *SRM* between the directrix and the focal distance of the point *R* in which it meets the directrix; and, in like manner, that the second tangent *RQ* from *R* bisects the supplementary angle SRN between SR and the directrix. Hence the two tangents to a parabola from any point on its directrix, or at the extremities of any focal chord, are at right angles; and conversely, the directrix of a parabola is the locus of the intersection of tangents at right angles.

Corollary 2.

To draw tangents to a parabola from any external point E, with centre E and radius ES describe a circle cutting the directrix in M and N, and let the diameters through those points meet the curve in P and Q, which will be the points of contact of the tangents required; since, as readily appears, EP bisects the angle SPM and EQ bisects the angle SQN.

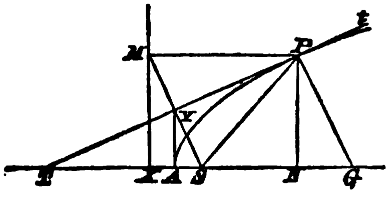
Corollary 3.

The subtangent at any point is double of the abscissa; since, in the next figure, $ST = SP = NX = AN + AS$, and therefore $AN = ST - AS = AT$, or NT the subtangent is equal to $2AN$.

PROPOSITION VII.

26. *The normal at any point of a parabola bisects the interior angle between the diameter and the focal distance of the point.*

If the normal at P meet the axis in G, then, by Art. 19,



$SG = SP$; and therefore PG makes equal angles with SP and the axis, and bisects the angle which SP makes with the diameter through P.

The same might have been deduced as a corollary from the preceding proposition.

Corollary.

If AN be the abscissa of P, then since $SG = SP = NX$, therefore $NG = NX - SN = SX = 2SA$, or the subnormal is equal to the semi-latus rectum.

PROPOSITION VIII.

27. *The tangent at the vertex of a parabola is the locus of the foot of the focal perpendicular upon the tangent at any point; and the focal perpendicular is a mean proportional to the focal distances of the vertex and of the point of contact of the variable tangent.*

(i) Let the diameter at any point P of a parabola be produced to meet the directrix in M, and let the tangent at A meet SM in Y.

Then, SY being evidently equal to MY, and SP being equal to PM, and PY common to the triangles SPY, MPY; therefore PY is at right angles to SM, and it bisects the angle SPM, and is therefore the tangent at P.

Hence Y, which by construction lies on the tangent at A, is the foot of the focal perpendicular upon the tangent at P; and conversely, the locus of the foot of the focal perpendicular on the tangent at P is the tangent at A.

This suggests an obvious method of drawing a second tangent to a parabola from a given point on the tangent at its vertex.

(ii) Again, since the two tangents from Y to the parabola subtend equal angles at S, the right-angled triangles SAY, SYP are similar, so that

$$SA : SY = SY : SP,$$

or

$$SY^2 = SA \, . \, SP.$$

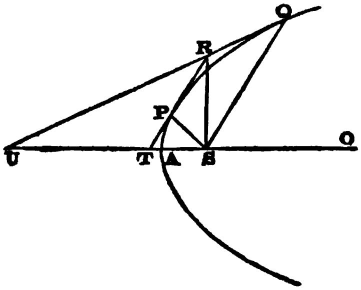
Corollary.

Since (Art. 24, Cor.) any two intersecting tangents to a parabola are in the subduplicate ratio of the focal distances of their points of contact, they are in the same ratio as the focal perpendiculars upon them.

PROPOSITION IX.

28. *The exterior angle between any two intersecting tangents to a parabola is equal to the angle which either of them subtends at the focus; and the inclination of either tangent to the axis is equal to that of the other to the focal distance of their common point.*

(i) Let the tangents at P and Q intersect in R, and meet the axis in T and U; and let O be a point in AS produced.



Then the exterior vertex angle PSO of the isosceles triangle PST being double of the interior base angle STP, and the angle QSO in like manner being double of SUQ, therefore by subtraction, in the figure drawn,

$$\angle PSQ = 2\,TRU.$$

Therefore, since the two tangents subtend equal angles at S, the angle subtended by either is equal to the exterior angle TRU between them.

Hence $\angle TRU$ will be acute or obtuse according as the focus lies without or within the segment of the curve cut off by PQ. In either case it will be seen that the *acute angle* between the tangents is equal to half the angle which their chord of contact subtends at the focus.

(ii) Since the angle TRU is equal to PSR, therefore

$$\begin{aligned}\angle SRU = PSR + PRS &= SPT\\ &= STP;\end{aligned}$$

or the angles which QR makes with SR are equal to those which TR makes with the axis.

Hence also, subtracting the angle TRU,

$$\angle SRP = STP - TRU = SUQ$$
$$= SQU.$$

Corollary 1.

If from any point R on the tangent at a fixed point P the second tangent RQ be drawn, the angle SRQ will be *constant*, since the equal angle SPR is fixed. If one of the tangents be the tangent at the vertex, SR will be the focal perpendicular upon the other.

Corollary 2.

The triangles SPR, SRQ are similar, having their angles at S equal, and likewise those opposite to SP and SR respectively. Hence $SR^2 = SP.SQ$, or the focal distance of the intersection of any two tangents to a parabola is a mean proportional to the focal distances of their points of contact; and each tangent is to the other as SR to the focal distance of the point of contact of the latter.

Corollary 3.

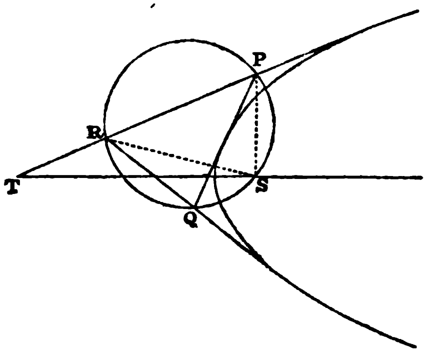
If two fixed tangents be cut by any third in points P and Q, as in the next figure, the triangle SPQ will have its angles constant, since, by Cor. 1, its angle at P is constant, and likewise its angle at Q. Again, in the same figure, if the three tangents be fixed, and if any fourth cut them in points L, M, N, then, the angles of the triangles SLN, SMN, being constant by the former case, the ratio of LN to MN is constant. Conversely, the envelope of a straight line which is cut in a constant ratio by three fixed straight lines is a parabola touching the three fixed lines.

PROPOSITION X.

29. *The circumscribed circle of any triangle whose three sides touch a parabola passes through the focus.*

Let PQR be any triangle whose three sides touch a parabola, and let PR meet the axis in T. Then, by Art. 28,

$$\angle SRQ = STP = SPQ.$$



Therefore the points S, P, R, Q are concyclic; that is to say, the focus S lies on the circumscribed circle of the triangle PQR.

Corollary 1.

Let p, q, r* be the points of contact of the tangents QR, RP, PQ, and let PQ meet the directrix in D, so that $\angle rSD$ is a right angle. Let the perpendicular drawn from R to PQ meet the directrix in O, and SD in N. Then the angles PQR and QSr are equal by Prop. IX., and therefore their complements are equal, so that

$$\angle QRN = QSN,$$

or N lies on the circle QRS, which also circumscribes the triangle PQR. Moreover, PQ bisects the angle ODN (Art. 25), and therefore also the line ON, to which it is at right angles. Hence O is the orthocentre of the triangle PQR, or *if any parabola be inscribed in a triangle, its directrix will pass through the orthocentre.*

Corollary 2.

If four tangents to a parabola be given, its *focus* is determined by the intersection of the circumscribed circles of any two of the triangles formed by the four tangents, and its *directrix* is the straight line joining the orthocentres of any two of them. Hence it appears that one parabola can in general be described touching four given straight lines.

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* See the lithographed figure, No. 2.

Corollary 3.

Since (Art. 13, Cor.) RQ and Pq subtend equal angles at S; and since, in the circle, $\angle SPq = SQR$; therefore the triangles SPq, SQR are similar, so that

$$QR : Pq = SR : Sq.$$

Hence, and by Art. 28, Cor. 2,

$$QR : Pq = Rp : Rq = Qp : PR;$$

or, if two tangents to a parabola be cut by any third, their alternate segments will have the same ratio, and this ratio will be *constant* if the two tangents be fixed.

SCHOLIUM.

The above proposition, with several deductions therefrom, is found in Section I. §§ 15—24 of I. H. LAMBERT'S *Insigniores Orbitæ Cometarum Proprietates* (Augustæ Vindelicorum, 1761). The proposition itself, together with Cor. 1, may be applied to prove certain properties of the straight line and circle, as below.

(i) In any given triangle, and with any point on its circumscribed circle as focus, suppose a parabola to be inscribed. Then, since the sides of the triangle are tangents to the parabola, the feet of the three focal perpendiculars upon them must lie on the tangent at the vertex. Hence, if from any point on the circumscribed circle of a triangle perpendiculars be let fall upon its three sides, the feet of the three perpendiculars will be collinear.

(ii) Supposing a parabola to be described touching four given straight lines, its focus must lie on the circumscribed circle of the triangle formed by any three of the said lines. Hence the circumscribed circles of the four triangles formed by any four straight lines meet in a point.

(iii) The directrix of the parabola touching four given straight lines passes through the orthocentres of the four triangles formed by those lines. Hence the orthocentres of the four triangles formed by any four straight lines are collinear.

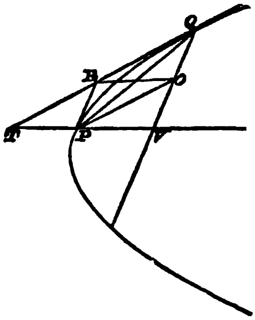
For the proof of Cor. 1 given above I am indebted to Mr. Rawdon Levett, of St. John's College, Cambridge. Another elementary proof, based upon the property that the feet of the focal perpendiculars on the three tangents are collinear, was given in No. 160, p. 63, of the *Lady's and Gentleman's Diary* (1863). The theorem in question, which is in reality a particular case of Brianchon's theorem (Salmon's *Conic Sections*, Art. 268), was propounded by J. STEINER in Crelle's *Journal für die reine und angewandte Mathematik*, vol. II. p. 191 (Berlin, 1827), and was demonstrated by him in Gergonne's *Annales de Mathématiques pures et appliquées*, vol. XIX. p. 59 (Paris, 1828), with the help of Pascal's theorem, as follows. If S be any point on a conic, PQR an inscribed triangle, and PP', QQ' chords through any point O; then in the hexagon $PP'SQ'QR$ the points ($P'S$, QR) and (SQ', RP) will lie on a straight line through O. If the conic be a *circle*, and O the *orthocentre* of the triangle, the straight line through O will evidently meet the continuations of the perpendiculars from S to the sides of the triangle at distances from S which are respectively double of those perpendiculars, and will therefore be the directrix of the parabola drawn with S as focus to touch the sides. Steiner himself likewise applied his theorem as in § iii. (Crelle, II. 97; Gergonne, XIX. 59).

PROPOSITION XI.

30. *The portion of any diameter intercepted by any tangent and the ordinate of its point of contact with respect to that diameter is bisected at the curve.*[*]

Let the diameter at P be met by the tangent at Q in T, and by the ordinate of Q in V; and let the tangent at P meet that at Q in R.

Complete the parallelogram $QRPO$ by drawing PO parallel to RQ. Then the diagonal RO bisects the diagonal PQ, which is also the chord of contact of the tangents RP, RQ. Therefore RO is a diameter of the parabola, and hence, all diameters being parallel,



$$PV = RO = PT,$$

or VT is bisected at P.

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[*] This is included in Art. 23, Cor. 3. See also Art. 18, Cor., and Art. 25, Cor. 3.

Corollary 1.

Any triangle whose base is parallel to the axis of a parabola has its remaining sides in the ratio of the parallel tangents; for supposing those sides parallel to the tangents in the figure, their ratio will be that of RP to RT, whereof the latter is equal to RQ.

Corollary 2.

If from any external point R there be drawn a tangent meeting the curve in P, and a chord meeting the curve in M, N, and the diameter through P in V, then by Cor. 1, and by Art. 16, Cor. 1, it is easily shewn that $RV^2 = RM.RN$.

Corollary 3.

It may be deduced from the proposition that the intercepts upon any diameter made by any two tangents and the ordinates of their points of contact are equal; and hence, that the area between the two tangents and the diameter is equal to half the area between their chord of contact, the ordinates of its extremities, and the diameter; and hence, that the triangle made by any three tangents is equal to half the triangle formed by joining their points of contact.

QUADRATURE.

PROPOSITION XII.

32. *The area of the parabolic segment upon any chord as base is equal to once and one-third of a triangle having the same base and altitude.**

Take RR' as the base of the segment, and suppose it parallel to the tangent at P.

Let the diameters through R and through an adjacent point Q on the curve meet the tangent at P in M and O; and let the diameter through P meet RQ in T, and RR' in U;

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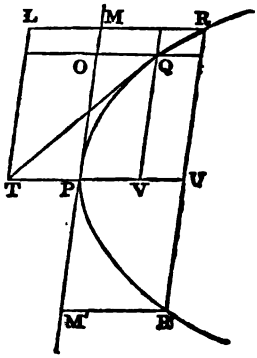
* This theorem, one of the great discoveries of ARCHIMEDES, was the first example of the exact quadrature by infinitesimals of a continuous curvilinear area. It forms the twenty-fourth and last proposition in his special treatise on the Quadrature of the Parabola. See the Oxford edition of his works, p. 33 (1792).

and let PV be the abscissa of Q. Complete the parallelogram $UTLR$ by drawing TL parallel to the base of the segment to meet RM produced.

Let Q coalesce with R, so that the chord QR becomes a tangent, and P becomes the middle point of VT, and therefore PM bisects the parallelogram QL.



Hence, and by Euclid I. 43,

the parallelogram $QU = QL = 2QM$.

Through any number of points on the arc PR draw parallels to RR' and PU, so as to form with PU two series of parallelograms, the one corresponding to QU and the other to QM, and let the number of the points be increased and their successive distances diminished indefinitely.

Then, as above, the several parallelograms in the former series become double of those in the latter, and the sum of the former, which is ultimately the parabolic area RPU, becomes double that of the latter, or of the parabolic area RPM.

Hence the semi-segment RPU is equal to two-thirds of the parallelogram MU, or to four-thirds of the triangle RPU; and the whole segment RPR' is equal to four-thirds of the triangle RPR', which has the same base and altitude.

Corollary.

Let the tangents at R, R' meet in T. Then the area of the segment is equal to two-thirds of the triangle formed by these tangents and its base, or the portion of the triangle on the concave side of the arc is double of that on its convex side.

This might have been proved by connecting the points R, R' by an infinity of consecutive chords, drawing the tangents at their extremities, and shewing, after the manner of Prop. XI., Cor. 3, that the area between the successive chords and RR' is double of that between the corresponding tangents and the tangents at R, R'.

EXAMPLES.

81. The radius of the circle through the vertex and the extremities of the latus rectum of a parabola is equal to five-eighths of the latus rectum.

82. A point on a parabola being given, if the focus also be given the envelope of the directrix will be a circle; or if the directrix be given the locus of the focus will be a circle.

83. If two parabolas have a common focus their common chord passes through the intersection of their directrices and bisects the angle between them.

84. The common chord of two parabolas which have a common directrix bisects the straight line joining their foci at right angles.

85. Deduce from Prop. I. that the ordinate of the middle point of a chord whose direction is given is of constant magnitude.

86. The perpendicular to a chord of a parabola from its middle point and the ordinate of that point intercept on the axis a length equal to the semi-latus rectum. Hence shew that the locus of the middle point of a focal chord, or of any chord which meets the axis in a fixed point, is another parabola.

87. Prove the following construction. Let AN be the abscissa of any point P on a parabola, and let MP be equal and parallel thereto. Divide NP into any number of equal parts and through the points of section draw parallels p_1, p_2, p_3... to the axis, and divide MP into the same number of equal parts in points 1, 2, 3.... Then will the lines p_1, p_2, p_3... meet $A1$, $A2$, $A3$... respectively on the parabola.

88. If PQ be a focal chord of a parabola, $SA.PQ = SP.SQ$.

89. If the ordinates or the focal distances of all points on a parabola be cut in a given ratio the locus of the points of section will in either case be a parabola.

90. Circles whose radii are in arithmetical progression touch a given straight line on the same side at a given point. If to each circle a tangent parallel to the given line be drawn it will cut the circle next larger in points lying on a parabola.

91. Find the locus of the centre of a circle which passes through a given point and touches a given straight line; or which touches a given circle and a given straight line.

92. If a parabola be made to roll upon an equal parabola, their vertices being initially coincident, the locus of the focus of the former will be the directrix of the latter.

93. Find the locus of a point which moves so that its shortest distance from a given circle is equal to its perpendicular distance from a given diameter of that circle.

94. The circle described on any focal chord of a parabola as diameter touches the directrix; and the circle on any focal radius touches the tangent at the vertex.

95. Given the focus, or the directrix, and two points of a parabola, shew how to construct the curve, and state the number of solutions in each case.

96. The diameters through the extremities of any focal chord of a parabola meet the chords joining them to the vertex upon the directrix and intercept upon it a length which subtends a right angle at the focus.

97. Two circles whose centres are on the axis of a parabola touch the parabola and one another. Prove that the difference of their radii is equal to the latus rectum.

98. Semicircles being described upon the segments of a focal chord, shew that the squares of their common tangents vary as the length of the chord.

99. The arms of any angle in a focal segment of a parabola meet the directrix at distances from the axis to which the semi-latus rectum is a mean proportional.

100. Shew how to place in a given parabola a focal chord of given length.

101. A parabola being given, find its axis, focus, vertex, and directrix.

102. If a chord be drawn to a parabola from the foot of its directrix, the rectangle contained by its segments will be equal to the rectangle contained by the segments of the parallel focal chord.

103. If AQ be a chord drawn from the vertex A of a parabola, and QR be a perpendicular to it at its extremity Q meeting the axis in R, then will AR be equal to the focal chord parallel to AQ.

104. If PQ be a focal chord of a parabola, and R any point on the diameter through Q, then will $\frac{PR^2}{PQ}$ be equal to the focal chord parallel to PR.

105. Find the locus of the points which divide parallel chords of a parabola into segments containing a constant rectangle.

106. The latus rectum is a mean proportional to the ordinates of the extremities of any chord which passes through the focus or through the foot of the directrix; and the rectangle contained by the abscisses of the extremities of the chord is equal to the square of the focal distance of the vertex.

107. If a chord subtends a right angle at the vertex, shew that it passes through a fixed point on the axis, and that the latus rectum is a mean proportional to the ordinates, and likewise to the abscisses, of its extremities.

108. Shew that the absciss cut off by any chord from any diameter is a mean proportional to the abscisses of its extremities with respect to that diameter, and that the corresponding ordinates are proportionals.

109. The distances from the focus at which the straight lines joining the ends of a focal chord to the vertex meet the latus rectum are alternately equal to the ordinates of the ends of the chord.

110. In a given parabola inscribe, and about it circumscribe, a triangle whose sides shall be parallel to three given straight lines.

111. A chord of a parabola and the chord joining the two points on the curve at which it subtends right angles intercept on the axis a length equal the latus rectum.

112. Deduce from Ex. 54 that the intercept on the axis made by any polar and the ordinate of its pole is bisected at the vertex.

113. The intercepts upon any diameter by any two polars and the ordinates of their poles with respect to that diameter are equal.

114. On a chord through a fixed point O a mean proportional OM is taken to the segments of the chord. Shew that the locus of M is a diameter.

115. A circle being described on a chord of a parabola which is parallel to a given line, shew that its centre is at a constant distance from the middle point of its opposite chord of intersection with the parabola.

116. If a circle cut a parabola in four points the ordinates of the points of section on one side of the axis will be together equal to the ordinate or ordinates of the point or points of section on the other side.

117. If three of the points of section coalesce their common ordinate will be equal to one-third of the ordinate of the fourth point; and the common chord of the circle and the parabola will be equal to four times their common tangent measured from its point of contact to the axis.

118. Three chords of a parabola drawn at right angles to a focal chord through its extremities and the focus are proportionals.

119. If from the vertex of a parabola a pair of chords be drawn at right angles, find the locus of the further vertex of the rectangle of which they are adjacent sides.

120. If a chord perpendicular to the axis be produced to meet the tangent at an extremity of the latus rectum, the rectangle contained by its segments will be equal to the square of its distance from the focus.

121. A chord of a parabola drawn from a given point on the curve is intersected by any ordinate of the diameter through that point and by the diameter through the extremity of the ordinate. Shew that the chord is a third proportional to its segments estimated from the given point to the ordinate and the diameter through its extremity respectively.

122. The ordinate of a point Q on the curve being intersected by its diameter in V, by any other diameter in R, and by the straight line joining the vertices of those diameters in R', shew that

$$QV^2 = VR . VR'.$$

123. The straight lines joining any point on a parabola to the extremities of a given chord meet any diameter at distances from its extremity which have the same ratio as the segments into which it divides the chord.

124. If from the point of contact of any tangent straight lines be drawn to two points on the curve, each to intersect the diameter through the other point, the two points of intersection will lie on a parallel to the tangent.

125. If the diameter through any point P of a parabola meet a given chord in a, and the tangents at its extremities in b, c, shew that $Pa^2 = Pb . Pc$, and deduce the theorem of Ex. 73* for the case of the parabola.

126. Three fixed points and a variable point being taken on a parabola, shew that the chords joining the latter to two of the fixed points cut off abscisses from the diameter through the remaining fixed point which are in a constant ratio.

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* In the example referred to, for "is" read *varies as*.

127. Shew that the preceding theorem is a special case of Ex. 80.

128. If a parabola which bisects the sides of a triangle *ABC* meet its sides again in *a*, *b*, *c*, then will *Aa*, *Bb*, *Cc* be parallels.

129. Through a given point within a parabola draw a chord which shall be divided in a given ratio at that point.

130. Describe a parabola through four given points; or through three given points, and having its axis in a given direction; and shew that the latter is a particular case of the former.

131. Shew that a circle can be described touching any two diameters of a parabola and the focal radii to their extremities; and hence, that any two intersecting tangents to a parabola subtend equal angles at the focus.

132. The four points of intersection of two parabolas whose axes are at right angles lie on a circle, and the sums of the ordinates of their points of intersection on opposite sides of the axis of either are equal.

133. If *ACP* be a sector of a circle of which *CA* is a fixed radius, and if a circle be drawn to touch *CA*, *CP* and the arc *AP*, the locus of its centre will be a parabola.

134. If a circle and a parabola touch at one point and intersect in two others, the diameters of the parabola at the latter points will meet the circle again on a parallel to the tangent at the former.

135. If a straight line be drawn from a fixed point on a circle to bisect any chord parallel to the diameter through that point, find the locus of its intersection with the diameter through an extremity of the variable chord.

136. If the two tangents from any point on the axis of a parabola be cut by any third tangent, their alternate segments will be equal.

137. If OP, OQ be the tangents at P and Q to a parabola, and Pp, Qq chords parallel thereto, the distances of O from pq and PQ will be in the ratio of five to one.

138. If there be three tangents to a parabola, whereof one is parallel to the chord of contact of the other two, shew that the three tangents contain an area equal to half the area of the triangle whose vertices are at their points of contact, and apply this result to prove Prop. XII.

139. The locus of the vertex of a parabola which has a given focus and a given tangent is a circle.

140. If the tangents at P and Q intersect in R, the circle through P touching QR in R passes through the focus.

141. The tangent at any point meets the directrix and the latus rectum at equal distances from the focus.

142. A chord of a parabola being drawn through a given point, determine when the rectangle contained by its segments will be a minimum.

143. Two equal parabolas have the same axis and directrix, and from a point on one of them two tangents are drawn to the other; shew that the perpendicular from that point to the chord of contact of the tangents is bisected by the axis.

144. If a leaf of a book be folded so that one corner moves along an opposite side the line of the crease will envelope a parabola.

145. The three straight lines drawn through the points of intersection of three tangents to a parabola at right angles to their respective focal distances meet in a point.

146. The centre of the circle through any two points on a parabola and the pole of the straight line joining them lies upon the focal chord at right angles to the focal distance of the said pole.

147. Any two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

148. The portion of any tangent intercepted by the tangents at two fixed points subtends a constant angle at the focus. In what case will the subtended angle be a right angle?

149. If QV be the ordinate of any point Q on a parabola, and if the diameter bisecting QV meet the curve in P, then will VP meet the tangent parallel to QV at a distance from its point of contact equal to $\frac{2}{3}QV$.

150. The tangent from the vertex of a parabola to the circle round SPN, where PN is the principal ordinate of a point P on the curve, is equal to $\frac{1}{2}PN$.

151. The focal vectors to the points of contact of a common tangent to a parabola and the circle on its latus rectum as diameter are equally inclined to the axis. Express the distance between the points of contact in terms of the latus rectum.

152. Describe an equilateral triangle about a given parabola; and shew that the focal distances of its vertices pass each through the opposite point of contact, and that the centre of gravity of the triangle must lie upon a certain fixed straight line perpendicular to the axis.

153. The segments of the sides of a regular polygon circumscribing a circle subtend equal angles at the centre. State an analogous property of the parabola.

154. Find the envelope of a straight line which cuts the sides OA, OB of a given triangle OAB in points P, Q such that the rectangle $OP.OQ$ is equal to $AP.BQ$.

155. Find the envelope of the straight line connecting the feet of the perpendiculars let fall from any point of a parabola upon the axis and the tangent at the vertex.

156. If PQ be a chord at right angles to the axis of a parabola, the perpendicular from P to the tangent at Q will

cut off from the diameter at Q a length equal to the latus rectum.

157. A circle being drawn through the focus of a parabola to cut the curve in two points, compare the angles between the tangents to the parabola and the tangents to the circle at those points.

158. If the tangents to a parabola at P and Q meet in O, and if the diameter through O meet PQ in V, shew that $OP.OQ = 2OS.OV$.

159. If the tangents at P, Q intersect in O and meet the tangent at R in P', Q', then will OR pass through the intersection of PQ' and $P'Q$.

160. A parabola being inscribed in a triangle so as to bisect one of its sides, shew that the perpendiculars from the vertices of the triangle upon any tangent are in harmonical progression.

161. The vertex of a constant angle whose sides envelope a parabola traces a hyperbola having the same focus and directrix.

162. In Art. 29, if PB and QM be perpendiculars to QR and the directrix, O the point in which the perpendicular from Q to PR meets the directrix, and PP' a diameter of the circle, shew that

$$SQ : QM = QB : PQ = QO : PP',$$

and deduce Steiner's theorem.

163. To two parabolas which have a common focus and axis, two tangents are drawn at right angles. Shew that the locus of their intersection is a straight line perpendicular to the axis; and examine the case in which the directrices of the two parabolas coincide.

164. Chords of a parabola being drawn to touch an equal parabola having the same vertex, their axes being in opposite directions, shew that the locus of the middle points of the chords is a parabola, whose linear dimensions are one-third of those of the original parabola.

165. Two parabolas have a common vertex, and their axes are in opposite directions. If the latus rectum of one of them be eight times that of the other, the intercept on any tangent to the former made by their common tangent and the axis will be bisected by the latter.

166. If the vertex of an angle of constant magnitude move on a fixed straight line, and one of its arms pass through a fixed point, the other will envelope a parabola of which the fixed point and line are the focus and a tangent.

167. If a focal chord meet any tangent at a given angle, determine the locus of their point of intersection.

168. If the tangents to a parabola at points P and Q intersect in O and meet the tangent at any point R in P' and Q', and if OR meet PQ in Z, then

$$PZ : QZ = P'R^2 : Q'R^2.$$

169. The locus of the foot of the focal perpendicular upon a normal chord of a parabola is a parabola.

170. If PQ be a chord normal at P and parallel to the focal chord FF', then

$$PQ : FF' = SY : SA,$$

where A is the vertex and SY the focal perpendicular upon the tangent at P.

171. If from a given point on a parabola any two chords be drawn making equal angles with the normal at that point, the focal distances of their further extremities will contain a constant rectangle.

172. The intercept on any tangent made by the curve and the tangent at the further extremity of the normal at its point of contact is bisected by the directrix.

173. If T be the pole of a chord PQ normal at P, and AN be the abscissa of P, shew that

$$PQ : PT = PN : AN.$$

174. The perpendicular to a normal to a parabola at the point in which the normal meets the axis envelopes an equal parabola, and the focal vector of the point at which the normal is drawn meets the envelope at the point in which the perpendicular touches it.

175. The normals at the ends of a focal chord intersect upon its diameter, and the locus of their intersection is a parabola.

176. The normal, terminated by the axis, is a mean proportional to the segments of the focal chord to which it is at right angles.

177. The squares of the normals at the extremities of a focal chord are together equal to the square of twice the normal perpendicular to the chord.

178. The normal at any point is equal to the ordinate which bisects the subnormal at that point.

179. The locus of the centre of the circle circumscribing the triangle SYP, where SY is the focal perpendicular on the tangent at any point P, is a parabola.

180. All circles which have their centres on a parabola and touch the tangent at its vertex are cut orthogonally by a circle which touches the parabola at its vertex and whose diameter is equal to the latus rectum.

181. From a point on any double ordinate QQ' a perpendicular is drawn to its polar to meet the polar in M and the axis in N. Shew that M, N, Q, Q', and the point in which the polar meets the axis are concyclic with the focus.

182. The continued products of the focal vectors to any three points on a parabola and of those to the poles of the chords joining the three points are equal.

183. If the tangents at p, q, r intersect in P, Q, R, as in Art. 29, and if O be the point in which the diameter through r meets pq, shew that

$$pQ : QR = RP : Pq = Qr : rP = pO : qO.^*$$

Shew also that

$$PQ.PR = Qr.Rq; \quad QR.QP = Rp.Pr; \quad RP.RQ = Pq.Qp;$$

and $$PQ.QR.RP = Pq.Qr.Rp = Pr.Qp.Rq.$$

184. Prove that in general two parabolas† and any number of central conics can be drawn through four given-points; and that no two parabolas or other conics can intersect in more than four points; and that no two parabolas can touch one another in more than one point.

185. If one triangle can be inscribed in a given circle (or ellipse) so that its three sides touch a given parabola, shew that any number of triangles can be so inscribed, and that the locus of their centroids is a straight line.

186. Any number of parabolas being described with the same vertex and axis, the polars with respect to them of all points on a fixed ordinate to the axis will meet in a point.

187. If a polygon be described about a parabola the continued products of the abscissæ of its vertices and of its points of contact respectively will be equal.

188. If T be the point of concourse of the tangents to a parabola at P and Q, and if p, q be the points in which any third tangent intersects them, then

$$\frac{Tp}{TP} + \frac{Tq}{TQ} = 1.$$

189. If from any point on the chord of contact of any two tangents to a parabola parallels to them be drawn each to intersect the other tangent, the points of intersection will lie

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* This is proved by APOLLONIUS in Lib. III., Prop. 41, of his Conics.

† Two chords of a parabola being given, it may be deduced from Art. 30, Cor. 2, that there are two possible directions of its axis.

on the tangent at the extremity of the diameter through the assumed point.

190. If P and Q be any two points on a parabola, and if PM, QN be the principal ordinates of P, Q, and AL the principal abscissa of the pole of PQ, shew that $PM.QN = 4AS.AL$.* Shew also that if O' be the pole of any chord drawn through any point O, and $O'V'$, OV be the ordinates of O' and O with respect to the diameter at any point P on the curve, then

$$OV.O'V' = 2SP(PV + PV').$$

191. If two parabolas be described each touching two sides of an equilateral triangle at the points in which it meets the third side, prove that they have a common focus and that the tangent to either of them at their point of intersection is parallel to the axis of the other.

192. If two parabolas be described each touching two sides of any triangle at the points in which it meets the third side, determine the area common to the two curves; and if three parabolas be so described, determine the area common to the three.

193. Any two tangents to a parabola intercept on two fixed tangents lengths which are in a constant ratio.

194. If P and Q be fixed points on a parabola, and RR' any double ordinate of a given diameter, then will RP and $R'Q$ meet that diameter at distances from the curve which will be in an invariable ratio.

195. The projections of any two tangents upon the directrix by lines radiating from the vertex are equal.

196. A triangle is revolving round its vertex in one plane; prove that at any instant the directions of motion of all the points of its base are tangents to a parabola.

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* This follows with the help of Examples 108 and 112, whereof the former may be deduced from Art. 80, Cor. 2, or from Prop. IV.

197. If three parabolas be inscribed in a given triangle, when will the area of the triangle formed by joining their foci be a maximum?

198. The area of the parabolic sector cut off by any two focal radii is equal to half the area bounded by the arc of the segment, the diameters through its extremities, and the directrix.

199. The difference of the ordinates of two points on a parabola being equal to QD, shew that the chord joining them will cut off a segment whose area is equal to $\frac{QD^3}{24AS}$. What is the envelope of a straight line which cuts off an area of given magnitude from a given parabola?

200. If the foci of four parabolas whereof each touches the straight lines joining the foci of the other three lie on a circle, the tangents at the vertices of the four parabolas will meet in a point.

CHAPTER IV.

CENTRAL CONICS.

In this chapter we shall deal with the common properties of the Central Conics, and in the next chapter with certain properties, viz. those of its asymptotes, which are peculiar to the Hyperbola.

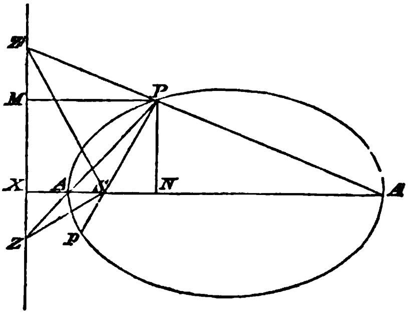
The *Abscissæ* or *Abscisses* of any point with respect to any diameter of a central conic are the segments of that diameter made by the ordinate of the point; and the *Central Abscissa* is the distance of the foot of the ordinate from the centre of the conic.

THE ORDINATE.

PROPOSITION I.

33. *The square of the principal ordinate of any point on a central conic varies as the rectangle contained by its abscissæ.*

Let the straight lines connecting the vertices A, A' of a central conic with any point P on the curve meet the directrix in Z and Z';* and let PN be the ordinate of P, and X the point in which the directrix meets the axis.



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* For the hyperbola, use the figure on p. 80, supplying the lines PAZ, $PZ'A'$.

Then (Art. 12) the intercept ZZ' subtends a right angle at S, and therefore $ZX.Z'X$ has the *constant* magnitude SX^2.

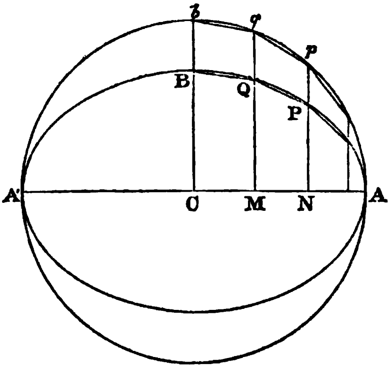
$$\begin{array}{lllll} \text{And since} & PN : AN & = ZX & : AX, \\ \text{and} & PN : A'N & = Z'X & : A'X; \\ \text{therefore} & PN^2 : AN.A'N & = ZX.Z'X & : AX.A'X \\ & & = SX^2 & : AX.A'X, \end{array}$$

which is an invariable ratio.

Let N be taken at the centre C of the conic, and let PN, in virtue solely of the above proportion, and without reference to the form of the curve, become equal to CB, so that

$$CB^2 : CA^2 = SX^2 : AX.A'X.$$

Then $PN^2 : AN.A'N = PN^2 : CA^2 \sim CN^2 = CB^2 : CA^2$.



In the ellipse it is evident that CB is equal to half the conjugate axis. In the hyperbola the conjugate axis does not meet the curve; nevertheless, for the sake of uniformity of expression, we shall define CB as the half of its length,* and the point B and a corresponding point B' equidistant from the axis as its *extremities*.

Corollary 1.

If the ordinate of P be divided in the ratio of the transverse to the conjugate axis at the point p, then $pN^2 = AN.A'N$.

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* The conjugate axis of any central conic is occasionally called its *Minor Axis*, although not necessarily less than the transverse axis, unless the curve be an ellipse.

Hence, when P lies on an ellipse, the locus of p is the circle described upon its major axis as diameter, and when P lies on a hyperbola, the locus of p is the hyperbola whose transverse axis is AA', and whose conjugate axis is equal to AA'.

Corollary 2.

If Pn be the ordinate of P with respect to the conjugate axis, it follows from the proposition that

$$PN^2 : CA^2 \sim Pn^2 = CB^2 : CA^2,$$

and

$$Pn^2 : CB^2 \sim Cn^2 = CA^2 : CB^2.$$

Hence also it may be shewn, after the manner of Cor. 1, that the locus of the point which divides Pn in the ratio of the conjugate to the transverse axis is either the circle on BB' as diameter, or the hyperbola whose transverse axis is BB', and whose conjugate axis is equal to BB'. Some of the uses of this corollary and the preceding will be pointed out in the chapter on Orthogonal Projection.

Corollary 3.

From Cor. 1 it appears, conversely, that if the ordinates of any number of points lying on the circle upon AA' as diameter be cut in any given ratio of minority $CB : CA$, the points of section will lie on an ellipse whose transverse axis is AA', and whose conjugate axis is equal to $2CB$. The smaller CB in comparison with CA, the less nearly circular is the ellipse; and ultimately, when CB vanishes, the ellipse becomes coincident with its major axis AA'. In like manner, the "complement"* of AA' is the limit to which the hyperbola described upon it as transverse axis tends when its conjugate axis is indefinitely diminished.

PROPOSITION II.

34. *The square of the ordinate of any point on a central conic with respect to any diameter is in a constant ratio to the rectangle contained by its abscissæ on that diameter.*

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* It may sometimes be convenient to speak of the remainder of an unlimited straight line from which any part has been taken away as the *Complement* of that part.

For by Art. 16, if QQ' be any chord parallel to a fixed straight line, and PP' be a fixed diameter meeting the chord in V, then the ratio $QV.VQ' : PV.VP'$ is constant. It follows as a special case that, if QQ' be a double ordinate of the diameter PP', then $QV^2 : PV.VP'$ is a constant ratio.

It is evident in the case of the ellipse that this result is equivalent to

$$QV^2 : PV.VP' = QV^2 : CP^2 - CV^2 = CD^2 : CP^2,$$

where CD is the semi-diameter parallel to the ordinate QV.

In the case of the hyperbola, supposing that CP meets and CD does not meet the curve, we might define the length of the semi-diameter CD and the position of its *extremity* D by the condition that CD^2 must be to CP^2 in the above-mentioned constant ratio, viz. that of the focal chords parallel to CD and CP, so that

$$QV^2 : PV.VP' = QV^2 : CV^2 - CP^2 = CD^2 : CP^2,$$

and therefore $\quad QV^2 + CD^2 : CD^2 = CV^2 : CP^2;$

but we shall at present merely remark that such a definition would be in accordance with the conventions usually adopted *

THE SECOND FOCUS AND DIRECTRIX.

PROPOSITION III.

35. *Every central conic has a second focus and directrix; and the sum of the focal distances of any point on the curve in the case of the ellipse, or the difference of the same in the case of the hyperbola, is constant and equal to the transverse axis.*

The existence of a second focus and directrix has been proved in Art. 14, Cor. 3; but it may also be deduced from the relation

$$PN^2 : CA^2 \sim Pn^2 = CB^2 : CB^2,$$

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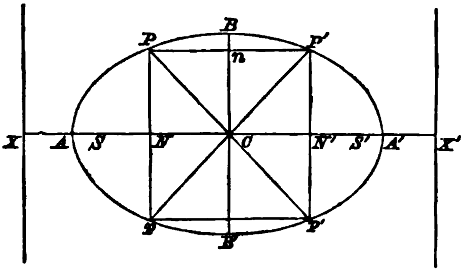
* Upon this subject, see Scholium C.

where PN and Pn are the ordinates of any point P on the curve with respect to its transverse and conjugate axes.

For since the ordinates with respect to either axis of the



points in which a parallel to that axis meets the curve are equal, and since, from the above proportion, the corresponding ordinates with respect to the other axis must consequently be equal, it is evident that points on a central conic may be determined in pairs as P, P' or p, p' on opposite sides of and equidistant from the one axis, and likewise in pairs as P, p or P', p' on opposite sides of and equidistant from the other axis.

The curve is therefore divided symmetrically by its conjugate axis as well as by its transverse axis, and it has a second focus S' equidistant with S from the centre, and a corresponding directrix meeting the axis at a point X' whose distance from the centre is equal to CX.

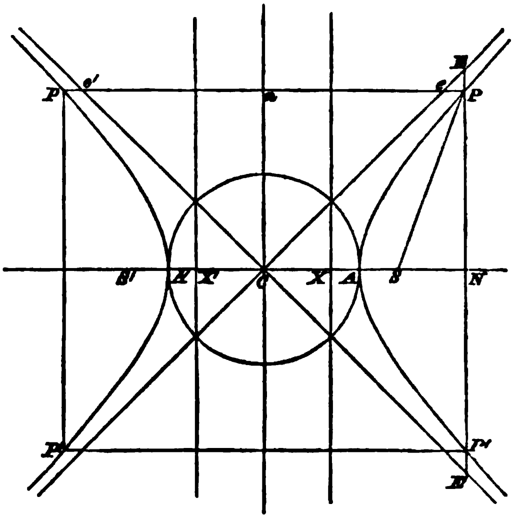
Hence, if P be any point on the curve,

$$S'P : NX' = S'A' : A'X' = SA : AX = SP : NX.$$

Therefore $$SP \underset{\sim}{+} S'P : NX \underset{\sim}{+} NX' = SA : AX.$$

Hence, in the ellipse, since $NX + NX'$ or XX' is constant, therefore $SP + S'P$ is equal to a constant length, viz. to $SA + S'A$ or AA'; that is to say, the sum of the focal distances of any point of the curve is equal to the major axis.

In the hyperbola, in like manner, $NX \sim NX'$ is constant, and therefore $SP \sim SP'$ is constant and equal to $SA \sim S'A$ or



AA'; that is to say, the difference of the focal distances of any point on the curve is equal to the transverse axis.

Corollary 1.

Since $SA : AX = SA' : A'X = \frac{1}{2}(SA \overset{+}{\sim} SA') : \frac{1}{2}(AX \overset{+}{\sim} A'X)$; and since the latter ratios are equal respectively to $CA : CX$ and $CS : CA$, or *vice versa*; therefore

$$CS : CA = CA : CX = SA : AX,$$

and

$$CS.CX = CA^2.$$

Corollary 2.

It may now be shewn, as in the next proposition, that

$$CB^2 = CS^2 \sim CA^2 = AS.A'S.$$

Hence, in the hyperbola, $CS^2 = CA^2 + CB^2$, or CS is equal to the distance AB between two adjacent extremities of the axes; and in the ellipse, $CS^2 + CB^2 = CA^2$, or SB is equal to the semi-axis major. In the case of the ellipse it follows more directly that, $SB = S'B = CA$; and hence, conversely, that

$$CB^2 = CA^2 - CS^2 = AS.A'S.$$

Corollary 3.

Hence $CS.SX = CS^2 \sim CS.CX = CS^2 \sim CA^2 = CB^2$,

and $SX : CX = CS.SX : CS.CX = CB^2 : CA^2$.

THE LATUS RECTUM.

PROPOSITION IV.

36. *The latus rectum is a third proportional to the transverse and conjugate axes.*

Since $CS : CA = SA : AX = SA' : A'X$;

therefore $CS + CA : CA = SA + AX : AX = SX : AX$,

and $CS \sim CA : CA = SA' \sim A'X : A'X = SX : A'X$.

Hence, and by Art. 33,

$$CS^2 \sim CA^2 : CA^2 = SX^2 : AX.A'X = CB^2 : CA^2,$$

and therefore

$$CS^2 \sim CA^2, \text{ or } AS.A'S, \text{ is equal to } CB^2.^*$$

Hence, by Prop. I., if SL be either of the ordinates corresponding to AS and $A'S$ as abscissæ,

$$SL^2 : AS.A'S = SL^2 : CB^2 = CB^2 : CA^2,$$

or the semi-latus rectum SL is a third proportional to CA and CB, and therefore the whole latus rectum is a third proportional to AA' and BB'.

Corollary.

If FF' be any focal chord, CD the parallel radius, and LL' the latus rectum, then by the proposition, and by Art. 34,

$$FF' : LL' = CD^2 : CB^2 = CD^2 : \tfrac{1}{2}LL'.CA,$$

or $$FF'.CA = 2\,CD^2;$$

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* The axis being a focal chord, it follows, from Art. 15, Cor., that $SL.CA$ is equal to $AS.A'S$, which in the ellipse may be shewn to be equal to CB^2. In the case of the hyperbola, the length CB may then be *defined* as a mean proportional to CA and SL. APOLLONIUS (Lib. III. 45) defined the points which we call the foci of central conics as certain points dividing the axis into segments whose rectangle is equal to $SL.CA$; but he nowhere mentions the focus of the parabola.

that is to say, any focal chord is a third proportional to the transverse axis and the diameter parallel to the chord.

SCHOLIUM A.

The Latus Rectum of the axis, according to Apollonius, was a certain straight line drawn at right angles to it from the vertex, equal to the double focal ordinate but defined without reference to the focus. The axis being regarded as Πλαγία, the *transverse* side of the "figure," the straight line drawn as above was taken as its 'Ορθία, or *erect* side; the term "figure" being used to denote the rectangle contained by those lines.

If AL be the latus rectum, and if the ordinate PN in Prop. I. meet $A'L$ in Q, then

$$PN^2 : AN.A'N = AL : AA' = QN : A'N = QN.AN : AN.A'N.$$

It follows that the square of the ordinate PN is greater than the rectangle $AL.AN$ contained by the latus rectum and the abscissa in the case of the hyperbola, and less than the same rectangle in the case of the ellipse; and hence the name Hyperbola, which signifies *excess*, and the name Ellipse, which signifies *deficiency*. The Parabola was so called from the *equality* of the square of the ordinate of any point upon it to the rectangle contained by its abscissa and the latus rectum. The names of the three conics have indeed been otherwise explained, but the interpretations of them given above are in accordance with the manner in which Apollonius introduces them. See Halley's edition of his works, Lib. I., props., XI., XII., XIII., pp 31—37 (Oxon., 1657). Moreover, it is reported by Proclus in his Commentaries on the first book of Euclid, at the commencement of the fourty-fourth proposition, upon the authority of "the Familiars of Eudemus," that the terms parabola, hyperbola, and ellipse had been used by the Pythagoreans to express the equality or inequality of areas, and were subsequently transferred to the conic curves for the reason given above. The passage is quoted in the original Greek on page 13 of E. F. August's *Zur Kenntniss der geometrischen Methode der Alten in besonderer Beziehung auf die Platonische Stelle im Meno 22d* (Berlin, 1843), and it may be seen in English in Thomas Taylor's translation of the Commentaries of Proclus, Vol. II. p. 198 (London, 1789). The whole work in the original Greek was printed at the end of *editio princeps* of Euclid's Elements (ed. Simon Grynæus, Basil., 1533); and it has also been edited separately by Godfr. Friedlein (Teubner, Leipzig, 1873).

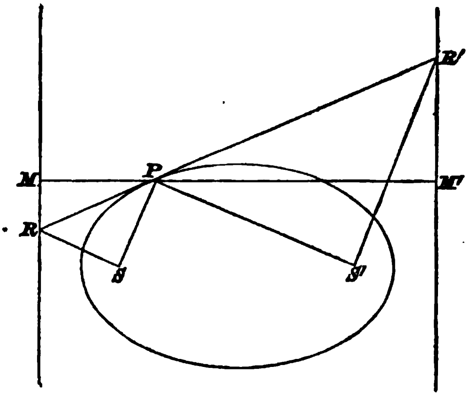
More generally, the *Latus Rectum* of any diameter was a length measured from its extremity upon the tangent thereat, equal in the case of the parabola to the parallel focal chord (although defined without reference to the focus), and in other cases a third proportional to the said diameter and its conjugate. Some later writers, as Mydorge, used the term *Parameter* for Latus Rectum in all cases, that of the axis being distinguished as *Recta Parameter*. Hamilton gave the alternative, "Latus Rectum sive Parameter istius diametri, &c."; but these expressions are now seldom used in geometrical treatises otherwise than as defined above on pages 1 and 44.

THE TANGENT.

PROPOSITION V.

37. *The tangent at any point of a central conic makes equal angles with the two focal distances of that point.*

Let the tangent at any point P to an ellipse, or a hyperbola, whose foci are S and S', meet the directrices in R and R' and let a parallel through P to the axis meet the directrices in M and M'.



Then since

$$SP : SP' = PM : PM' = PR : PR',$$

and since PR and PR' subtend right angles at S and S' respectively, therefore the triangles SPR, $S'PR$ are similar, having their angles at P equal; that is to say, the tangent at P makes equal angles with SP and $S'P$.

In the ellipse, the tangent lies without the angle SPS' and bisects it *externally*.

In the hyperbola (fig. p. 80), the tangent must cut the axis between A and A' (since otherwise it could not lie wholly without the curve), and therefore between S and S'; it must therefore bisect the angle SPS' *internally*.*

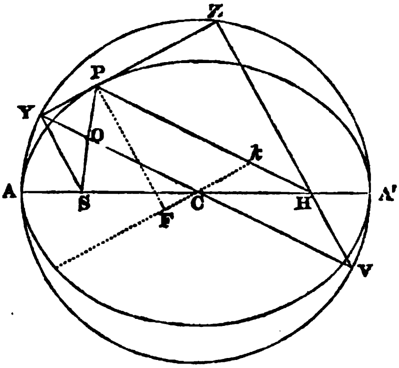
Corollary.

If P be one of the four points of intersection of an ellipse and a hyperbola which have the same foci S and S', their tangents at P, being the two bisectors of the angle SPS', will be at right angles to one another. Hence *confocal conics intersect at right angles.*

PROPOSITION VI.

38. *The projections of the foci upon the tangent at any point of a central conic lie on its auxiliary circle; and the semi-axis conjugate is a mean proportional to the distances of the foci from their respective projections.*

Let S and H be the foci, Y and Z their respective projections upon the tangent at any point P. Then will Y and Z



lie on the circle described upon the axis AA' as diameter, and the rectangle $SY.HZ$ will be equal to CB^2.

(i) For let a parallel to HP through Y meet SP in O. Then the tangent makes equal angles with SP and the parallel to HP, viz. the angles OPY, OYP, and the complements of these angles, viz. OSY, OYS are equal, therefore

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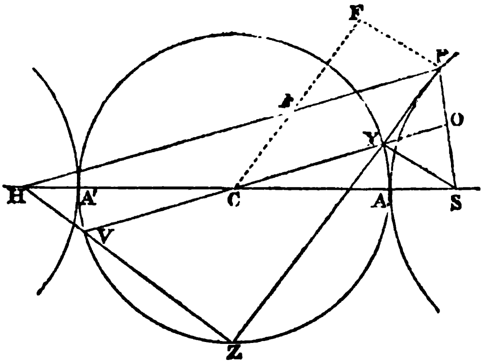
* If, in accordance with the principle of the note on p. 22, the distance of any point P on a hyperbola from S' be estimated *within* the curve, in which case it will be the "complement" of $S'P$, we may say of the hyperbola, as of the ellipse, that the tangent bisects the angle between the focal distances *externally*.

$$OP = OY = OS.$$

Hence, the parallel to HP bisects SH, viz. at the centre C of the conic, and



$$CY = CO \pm OY = \tfrac{1}{2}HP \pm \tfrac{1}{2}SP = CA,^*$$

the upper signs being taken in the case of the ellipse, and the lower in the case of the hyperbola.

Hence, the auxiliary circle is the locus of the foot of the perpendicular from either focus to the tangent at any point of the conic; and, conversely, the straight lines drawn from any point on the auxiliary circle at right angles to the two focal distances of that point are tangents to the conic.

(ii) Let ZH meet the circle again in V; then, the angle at Z being a right angle, VY passes through the centre C.

Hence, evidently SY, is equal to HV; and therefore,

$$SY.HZ = HV.HZ = HA.HA' = CB^2,$$

or CB is a mean proportional to the focal perpendiculars upon any tangent.†

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[*] This result might also have been arrived at by supposing SY, HP to meet in a point S', and shewing that $CY = \tfrac{1}{2}HS' = CA$.

[†] In the figure on p. 88, if T be the point of concourse of *any* two tangents, then $SY : HZ' = SY' : HZ$. It follows that angle $STY = HTZ'$; or, conversely, from Prop. XVIII. it may be deduced that $SY \,.\, HZ = CB^2$.

Corollary 1.

This proposition enables us to draw a tangent to the conic which shall be parallel to any given straight line, viz. by drawing SY (to a point Y on the auxiliary circle) at right angles to the given straight line, and drawing YP at right angles to SY.

Corollary 2.

The points V, Z in which any focal chord meets the auxiliary circle lie upon parallel tangents to the conic, and the semi-axis conjugate is a mean proportional to the perpendiculars HV, HZ from either focus H upon any two parallel tangents.

Corollary 3.

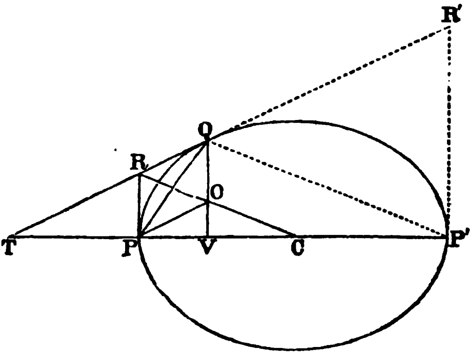
The diameter parallel to the tangent at P intercepts on either focal distance HP a length Pk equal to CY or CA.

PROPOSITION VII.

39. *The distance from the centre at which any tangent meets a given diameter varies inversely as the central abscissa of its point of contact with respect to that diameter.*

Let the tangent at any point Q meet a given diameter in T, and let CV be the abscissa of Q with respect to that diameter; then will the rectangle $CV.CT$ be of constant magnitude.

(i) Let CT *meet the curve* in P, and let the tangents at P and Q intersect in R; complete the parallelogram $QRPO$.



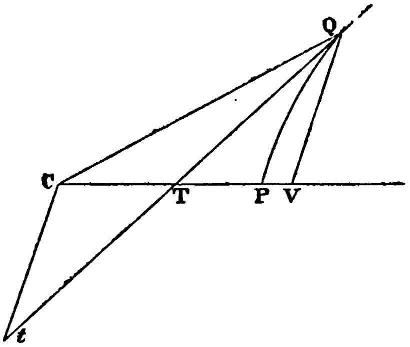
Then since, in the parallelogram, RO bisects PQ, and since PQ is the chord of contact of the tangents from R, therefore RO passes through the centre C.

Hence, by parallels,

$$CV : CP = CO : CR = CP : CT,$$

or $CV.CT$ has the constant magnitude CP^2, and CT varies inversely as the abscissa CV.

(ii) Next, let t be the point of concourse of QT with any diameter of a hyperbola *which does not meet the curve;* and let the ordinate QV in the former case of the proposition be supposed an ordinate of the conjugate diameter, and therefore parallel to Ct.



Then since $\qquad Ct : CT = QV : VT,$

therefore $\qquad QV.Ct : CV.CT = QV^2 : CV.VT.$

And since, by the former case, $CV.CT$ is equal to CP^2, therefore

$$QV.Ct : CP^2 = QV^2 : CV^2 - CP^2;$$

and therefore (Art. 34) $QV.Ct$ is equal to the square of the "semi-diameter" CD parallel to QV; that is to say, Ct varies inversely as QV, or as the abscissa of Q with respect to the diameter on which Ct is estimated.

Corollary 1.

The relation $CV.CT = CP^2$ implies that any diameter PP' which meets the conic is divided harmonically (Art. 18, Cor.)

at V and T. This likewise involves the equality of the rectangle $TC.TV$ to $TP.TP'$, which may also be deduced independently from the evident parallelism of $P'Q$ and CR, thus

$$TP : TV = TR : TQ = TC : TP'.$$

Or again, supposing the tangents at Q and P' to meet in R', we might have inferred from Art. 16, Cor. 1, that

$$RP : RQ = R'P' : R'Q,$$

and thence that TP, TV, TP' are in harmonic progression.

Corollary 2.

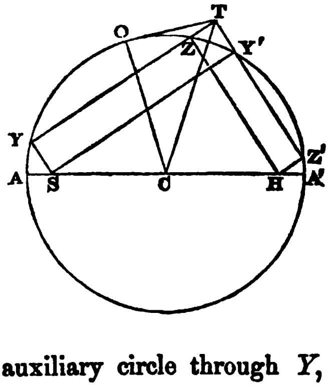
If CV and CT be estimated on the transverse axis, their product will be equal to CA^2; or if on the conjugate axis, it will be equal to CB^2.

THE DIRECTOR CIRCLE.

PROPOSITION VIII.

40. *The locus of the vertex of a right angle whose sides envelope a central conic is a circle.*

Let T be the point of concourse of a pair of tangents at right angles; Y and Z the projections of the foci upon one of them; Y' and Z' their projections upon the other.



(i) Draw the auxiliary circle through Y, Z, Y', Z'; and first, if T lie without it (that is to say, in the case of the ellipse), let TO be drawn touching it in O.

Then

$$TO^2 = TY'.TZ' = SY.HZ = CB^2,$$

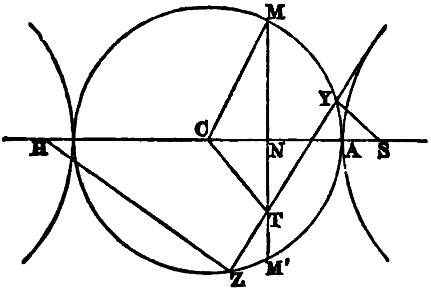
and therefore $CT^2 = CO^2 + CB^2 = CA^2 + CB^2$,

or the locus of T is a circle concentric with the conic, and whose diameter is equal to the diagonal of the rectangle contained by the axes.

(ii) Next let T lie within the circle, as may happen in the case of the hyperbola. Then it may be shewn in like manner that

$$CA^2 - CT^2 = TY.TZ = CB^2,$$

or $$CT^2 = CA^2 - CB^2.$$



Hence, provided that the transverse be greater than the conjugate axis, the locus of T will be a circle, the square of whose radius is equal to $CA^2 - CB^2$; but if the conjugate axis be the greater, the locus will be imaginary, or an "obtuse" hyperbola *can have no real tangents at right angles.*[*]

Corollary 1.

In like manner it may be shewn that, if the sides of a right angle envelope two *confocal* conics whose semi-axes are CA, CB and $C\alpha$, $C\beta$ respectively, its vertex will lie on a concentric circle the square of whose radius is equal to $CA^2 \pm C\beta^2$, or $C\alpha^2 \pm CB^2$. When $C\beta$ vanishes, one arm of the right angle passes through a focus, and its vertex Y (Art. 38) lies on the auxiliary circle. In the proposition itself, if CB be supposed to vanish, the director circle becomes the circle on SH as diameter.

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[*] In this case it will be seen that the theorem is applicable to the Conjugate Hyperbola. In the limiting case in which the axes are equal the locus reduces to a point, the only tangents at right angles being the Asymptotes.

Corollary 2.

A conic and its director circle are so related that any rectangle circumscribing the former is inscribed in the latter.

SCHOLIUM B.

The term DIRECTOR CIRCLE has been used of late years to denote the locus of intersection of tangents at right angles to a conic, with reference to the circumstance that when the conic degenerates into a parabola this locus becomes coincident with the directrix (Art. 25). The analogous term "Director Sphere" was introduced by Professor Townsend in the *Quarterly Journal of Pure and Applied Mathematics*, vol. VIII. p. 10 (1867). DE LA HIRE proved in his *Sectiones Conicæ* (PARIS, 1685), Lib. VIII. props. 27, 28, that the tangents to a conic from any point on a concentric circle whose radius is equal to $\sqrt{(CA^2 \pm CB^2)}$ meet the circle again at the ends of a diameter, and therefore contain a right angle. He also gave the equivalent of Cor. 2.

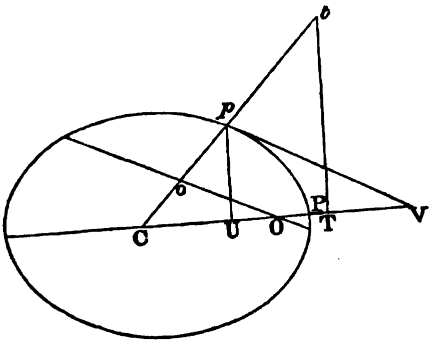
In some treatises the term Director Circle is used to denote the circle described about the further focus H with radius equal to the transverse axis, which possesses a property analogous to that of the directrix of the parabola; for if Y be any point on the circumference of the circle, and if HY meet the conic in P, then $SP = PY$, or the focal distance of P is equal to its normal distance from the circle. This circle affords a construction, analogous to that in Art. 25, Cor. 2, for drawing tangents to a central conic from any external point. Nevertheless it scarcely deserves a distinctive name; whereas the "director circle," according to the former definition, is of considerable importance in the higher geometry of conics. The analogy of the circle in question to the directrix of the parabola was pointed out by BOSCOVICH, (*Sectionum Conicarum Elementa*, § 102), but he did not give it a name.

POLAR PROPERTY.

PROPOSITION IX.

41. *The tangents at the extremities of any chord drawn through a given point intersect on a fixed straight line parallel to the ordinates of the diameter through that point.*

(i) Through a given point O within or without the conic draw any chord, and let the tangents at its extremities meet



in t. Draw Ct bisecting the chord in o, and meeting the curve in p, and let pU and tT be ordinates of the diameter through O.

Let CO meet the curve in P, and let it meet the tangent at p, which is parallel to oO, in V.

Then $$CO : CV = Co : Cp,$$

and $$CU : CT = Cp : Ct;$$

and therefore, the lengths Co, Cp, Ct being continued proportionals (Art. 39),

$$CO.CT = CU.CV = CP^2.$$

Hence T is a fixed point, and tT, which was ordinately applied to the diameter CP, is a fixed straight line.

Conversely, the chord of contact of the tangents drawn to a conic from any point t on a fixed ordinate passes through a fixed point O situated on the diameter of that ordinate.

(ii) In the case of the hyperbola, either or both of the diameters Co, CO may not meet the curve.

If Co only do not meet the curve, let Pu be the ordinate of P with respect to it, and let the tangent at P meet it in v; then, $Co.Ct$ being equal to $Cu.Cv$ (Art. 39, § ii),

$$CO : CP = Co : Cu = Cv : Ct = CP : CT,$$

as in the first case. But if CO only do not meet the curve, the first proof is applicable as far as $CO.CT = CU.CV$; and it is known from Art. 39, § ii. that the latter rectangle is of constant magnitude.

In the case in which neither Co nor CO meets the curve, if Cp be taken a mean proportional to Co and Ct, and if pV be drawn parallel to oO,* it will be seen that $CU.CV$ is still constant, and tT will be a fixed straight line as in the cases previously considered.

THE NORMAL.

PROPOSITION X.

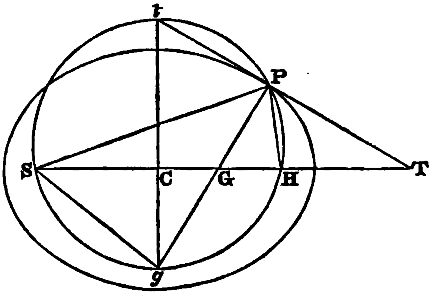
42. *The normal at any point of a central conic bisects the angle between the two focal distances of that point.*

If P be any point on a conic whose foci are S and H, it follows as a corollary from Prop. V. that the normal at P bisects the angle SPH in the case of the ellipse, and its supplement in the case of the hyperbola. The same may also be deduced from Art. 10 (where S may be either focus) as follows.

If the normal meet the axis in G, then

$$SG : SP = CS : CA = HG : HP;$$

and therefore PG bisects the angle SPH internally or externally, according as G lies in SH, as in the case of the ellipse, or in the "complement" of SH, as in the case of the



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* The line thus drawn is the tangent at p to the *Conjugate Hyperbola*, which will be defined in the next chapter. For another proof of the proposition, see Art. 17. The proof in Art. 41, § i. applies, with obvious modifications, to the Parabola.

hyperbola. In either case the normal bisects the angle between those portions of the focal distances which fall *within the curve.*

If the circle round SPH meet the conjugate axis in g and t, it is evident that Pg and Pt will be the two bisectors of the angle SPH; that is to say, they will be the tangent and the normal at P. This suggests an obvious method of drawing the two tangents or normals to a conic from any point on its conjugate axis.

Corollary 1.

The tangent and normal to a conic whose foci are S and H divide the straight line SH harmonically, and CS is a mean proportional to the lengths CG, CT which they intercept on the axis.

Corollary 2.

It is likewise evident by similar triangles that

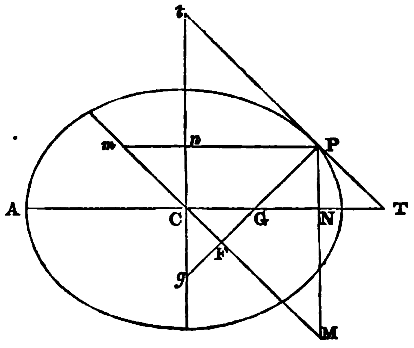
$$PG.Pg = PT.Pt = SP.HP;$$

and it will be shewn that each of these rectangles is equal to the square of the semi-diameter parallel to the tangent at P.

PROPOSITION XI.

43. *At any point of a central conic the normal, terminated by either axis, varies inversely as the central perpendicular upon the tangent, and directly as the radius parallel to the tangent.*

Let the tangent and the normal at P meet the transverse



axis in T and G, and the conjugate axis in t and g, respectively draw perpendiculars PN, Pn to those axes, and let them mee the diameter parallel to the tangent in M and m; and let the normal meet that diameter in F.

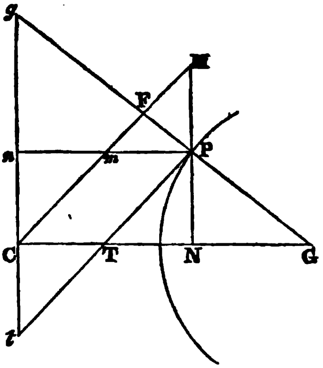
(i) Then (Art. 39, Cor. 2), the angles at N and F being right angles,

$$PG.PF = PN.PM = Cn.Ct = CB^2;$$

and, in like manner, the angles at n and F being right angles,

$$Pg.PF = Pn.Pm = CN.CT = CA^2;$$

that is to say, PG and Pg vary inversely as PF, which is equal to the central perpendicular upon the tangent at P.



(ii) It will be proved in the section on conjugate diameters that

$$PG : CD = CD : Pg = CB : CA,$$

where CD is the semi-diameter parallel to the tangent at P.

Corollary.

It is hence evident that

$$NG : CN = NG : Pn = PG : Pg = CB^2 : CA^2,$$

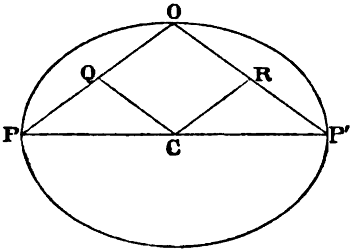
or *the subnormal varies as the abscissa.* In like manner it may be shewn that the subnormal ng on the conjugate axis varies as the abscissa Cn.

CONJUGATE DIAMETERS.

PROPOSITION XII.

44. *Supplemental chords are parallel to conjugate diameters.*

Let OP', OP be any two supplemental chords, and CQ, CR the diameters to which they are parallel; then will CQ, CR be conjugate. For it is evident that these diameters bisect OP and OP' respectively; that is to say, each of them bisects a chord parallel to the other.



This enables us to determine the relation between the directions of any two conjugate diameters, for, in Art. 33,

$$PN^2 : AN.NA' = CB^2 : CA^2,$$

where the supplemental chords AP, $A'P$ may be supposed parallel to any assumed pair of conjugate diameters. Hence, if the ratio of PN to AN (or the direction of one of the diameters) be given, the ratio of PN to $A'N$ (or the direction of the conjugate diameter) is determined.*

Corollary 1.

It readily follows that if P and D be points on two diameters of a central conic whose ordinates PN and DR (as in Art. 45) are so related that

$$PN : CR = DR : CN = CB : CA,$$

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* An equivalent result may be deduced from Art. 14, Cor 4 and Art. 35, Cor. 3. If θ and ϕ be the inclinations of two conjugate diameters to the axis, $\tan\theta \tan\phi = \mp \frac{b^2}{a^2}$ (where a, b are the semi-axes), the negative sign being taken for the ellipse and the positive for the hyperbola.

then will CP and CD be conjugate, provided that they lie in adjacent quadrants in the case of the ellipse, or in the same quadrant or in opposite quadrants in the case of the hyperbola. If the ordinates of P and D, in the case of the ellipse, be produced to meet the auxiliary circle in p and d, as in the next figure, *the angle pCd will a right angle*, as will be noticed more particularly in the chapter on Orthogonal Projection.

Corollary 2.

In the hyperbola it is evident that of every two supplemental chords one must lie wholly within and the other wholly without the curve; and hence that *one and one only of every two conjugate diameters meets the curve.*

Corollary 3.

To draw a pair of conjugate diameters inclined at a given angle, let a segment of a circle containing the given angle, and described on any diameter PP' as base, meet the conic again in O; then will the diameters parallel to OP and OP' be inclined at the given angle.

PROPOSITION XIII.

45. *The sum of the squares of any two conjugate diameters is constant in the ellipse, and the difference of the same is constant in the hyperbola.*

(i) If CP, CD be any two radii of an ellipse, and CN, CR the central abscisses of their extremities P and D respectively, then, by Art. 33,

$$PN^2 : CA^2 - CN^2 = DR^2 : CA^2 - CR^2 = CB^2 : CA^2.$$

Let CP and CD be supposed to lie in adjacent quadrants, and let

$$CN^2 + CR^2 = CA^2;$$

then the above proportions reduce to

$$PN : CR = DR : CN = CB : CA,^*$$

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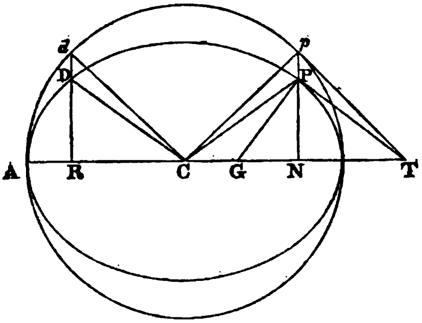
* The same proportions will hold when the ordinates are oblique, if CA and CB be the lengths of the corresponding semi-diameters.

and therefore (Art. 44, Cor. 1) CP, CD are conjugate, and conversely.



Hence also $$PN^2 + DR^2 = CB^2,$$

and $$CP^2 + CD^2 = CN^2 + PN^2 + CR^2 + DR^2 = CA^2 + CB^2;$$

or the sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes.

(ii) In the case of the hyperbola, let P be a point on the curve, and D a point on the hyperbola whose transverse and conjugate axes are BB' and AA' respectively; then, by Art. 33,

$$PN^2 : CN^2 - CA^2 = DR^2 : CR^2 + CA^2 = CB^2 : CA^2.$$

Let CP and CD be supposed to lie in the same quadrant or in vertically opposite quadrants, and let CD be regarded as *terminated* at the point D.*

Let $$CN^2 - CR^2 = CA^2;$$

then the above proportions reduce to

$$PN : CR = DR : CN = CB : CA,$$

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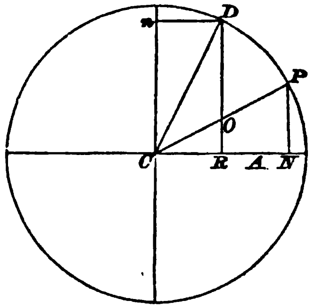
* The radius CD which does not meet the curve (Art. 44, Cor. 2) is here regarded as terminated by the Conjugate Hyperbola. For another proof of the proposition, in which the length of CD is defined as suggested in Art. 34, the reader is referred to the next chapter; and for a third proof, to the chapter on Orthogonal Projection. It will be seen that the above conventions with regard to the length of CD are consistent with one another, but the true definition of the lengths of diameters which do not meet the curve is that given in Scholium C.



and therefore (Art. 44, Cor. 1) CP, CD are conjugate, and conversely.

Hence also $$DR^2 - PN^2 = CB^2,$$

and $$CP^2 - CD^2 = CN^2 + PN^2 - (CR^2 + DR^2) = CA^2 - CB^2;$$

or the difference of the squares of any two conjugate "semi-diameters" of a hyperbola is constant and equal to the difference of the squares of the semi-axes.

Corollary 1.

Let the normal at P, which is at right angles to CD, meet the transverse and conjugate axes in G and g respectively; then it may be shewn by similar triangles that

$$PG : CD = PN : CR = CB : CA,$$

and $$Pg : CD = CN : DR = CA : CB.$$

Hence CD is a mean proportional to PG and Pg.

Corollary 2.

If P be any point on a conic whose foci are S and H, then since C is the middle point of SH,

$$2CP^2 + 2CS^2 = SP^2 + HP^2 = 4CA^2 \mp 2SP.HP.$$

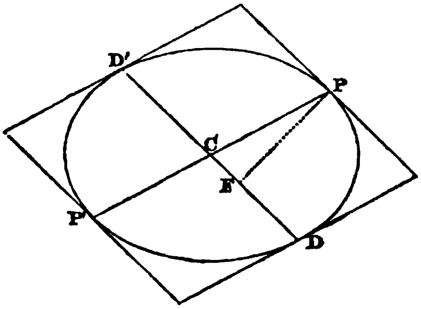
Hence $CP^2 \pm SP.HP = CA^2 + CA^2 - CS^2 = CA^2 \pm CB^2$,

and therefore, by the proposition, $SP.HP$ is equal to CD^2.

PROPOSITION XIV.

46. *The area of any parallelogram whose sides are equal and parallel to two conjugate diameters of a central conic is equal to the area of the rectangle contained by the axes.*

(i) Let a parallelogram be formed by drawing parallels to two conjugate diameters PP', DD' of a central conic through



their extremities, and let the normal at P meet DD' in F, and let it meet the axis in G.

Then since (Art. 45, Cor. 1),

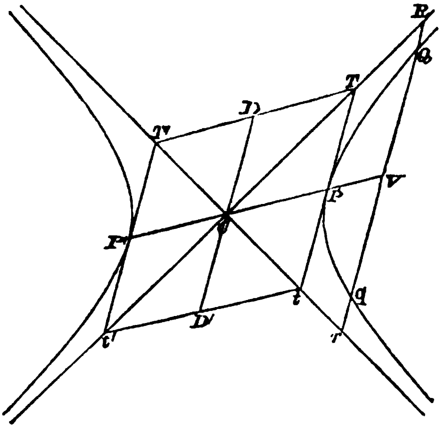
$$PG : CD = CB : CA,$$

therefore $$PF.PG : PF.CD = CB^2 : CA.CB$$

But, by Art. 43, the antecedents of this proportion are equal.

Therefore $$PF.CD = CA.CB,$$

and the area of the whole parallelogram is equal to $4PF.CD$, that is, to $2\,CA.2\,CB$, or to the rectangle contained by the axes.



In the case of the ellipse the parallelogram described as above completely envelopes the curve; but in the case of the hyperbola two of its sides only touch the curve (Art. 44, Cor. 2).

(ii) The proposition might also have been proved by shewing that, with the construction of Art. 45,

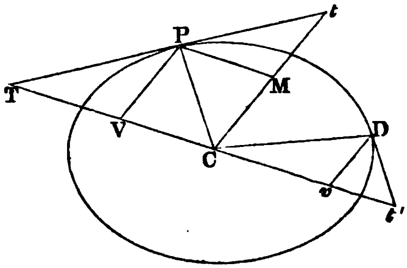
$$\Delta PCD = PDRN - PCN \mp DCR = \tfrac{1}{2}(CN.DR \pm CR.PN),$$

which last expression reduces to $\frac{1}{2}CA.CB$.

PROPOSITION XV.

47. *The intercept upon any tangent by any two conjugate diameters is divided at the curve into segments to which the parallel radius is a mean proportional.*

Let the tangent at P meet any two conjugate diameters in T and t, and let CD be the radius parallel to the tangent.



Let PV, Dv be ordinates of the diameter CT, and PM an ordinate of Ct; and let the tangent at D meet the former diameter in t'.

Then, by similar triangles, the tangent at D being parallel to CP,

$$PT : CT = CD : Ct',$$

and $$Pt : CV = Pt : PM = CD : Cv.$$

Therefore* $$PT.Pt : CV.CT = CD^2 : Cv.Ct';$$

and the consequents in this proportion being equal by Art. 39, therefore $$PT.Pt = CD^2,$$

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* More briefly, the condition that CT, Ct should be conjugate (note, p. 96) gives at once the relation $PT.Pt : CP^2 = CD^2 : CP^2$.

or CD is a mean proportional to PT and Pt; and when the point P is given the rectangle $Pt.PT$ is constant.

Corollary.

If the tangent at any point Q meet any two parallel tangents in R and R', then will the radius parallel to the tangent be a mean proportional to QR and QR'. For, in the figure of Art. 39, it is evident that CR and CR' are parallel to the supplemental chords $P'Q$, PQ, and are therefore conjugate diameters. Moreover, if RC meet $R'P'$ in r, the semi-diameter parallel to RP will be a mean proportional to $P'r$, $P'R'$, and therefore to PR, $P'R'$.

SCHOLIUM C.

Although the conjugate axis of a hyperbola and its other diameters which do not meet the curve are commonly regarded as terminated by the conjugate hyperbola, this convention is by no means accurate, but the true account of the matter is that given below.

Given the relation (Art. 33) between the coordinates of any point on the hyperbola

$$PN^2 : CN^2 - CA^2 = CB^2 : CA^2,$$

the true length of the semi-axis conjugate is found, by making CN vanish, to be $\sqrt{(-1)}\, CB$. Let this be denoted by $C\beta$, so that

$$PN^2 : CA^2 - CN^2 = -CB^2 : CA^2 = C\beta^2 : CA,$$

which shews that the hyperbola may be regarded as *an ellipse whose minor axis is a certain imaginary quantity.* In like manner the true length of the semi-diameter conjugate to CP, in the second case of Art. 34, is $\sqrt{(-1)}\, CD$; and if this be denoted by $C\delta$, we may write

$$QV^2 : CP^2 - CV^2 = C\delta^2 : CP^2,$$

as in the case of the ellipse. Now, treating β and δ as if they were real points on the curve and supposing ρ to be the projection of δ upon the axis, we have, precisely as in Art. 45, § i,

$$CN^2 + C\rho^2 = CA^2; \quad PN^2 + D\rho^2 = C\beta^2; \quad CP^2 + C\delta^2 = CA^2 + C\beta^2;$$

which will be seen to be equivalent to the results of Art. 45, § ii,

$$CN^2 - CR^2 = CA^2; \quad PN^2 - DR^2 = -CB^2; \quad CP^2 - CD^2 = CA^2 - CB^2.$$

And so in other cases (cf. Art. 40) we may pass at once to properties of the ellipse, in so far as they involve CB^2 and CD^2, by writing in place thereof $C\beta^2$ and $C\delta^2$, that is to say, *by changing the signs of CB^2 and CD^2.*

Next consider the hyperbola as a particular form of ellipse whose determining ratio has become one of majority. When this ratio has increased up to unity the further focus and vertex are at infinity; but as soon as it exceeds unity they at once come back from infinity to H and A' on the opposite of A. The true length of the axis of the hyperbola is therefore $A \infty A'$, the complement of AA', and the distance between its foci is $S \infty H$, the complement of SH. In like manner the distance of any point P on the curve from the further focus H is to be regarded as $H \infty P$, the complement of HP; and thus the fundamental bifocal property assumes the indeterminate form $SP + H \infty P = A \infty A'$. In order to pass to the determinate form of the bifocal property of the hyperbola from the property of the ellipse, we may remark that a point moving in a straight line from the further focus H of the former to a point P on S-branch of the curve may be supposed to move either (1) *within the curve*, from H to ∞ and from ∞ to P, always in the same direction, or (2) along the finite length HP *in the opposite direction;* and we may therefore regard HP, drawn towards the convexity of the S-branch, as essentially *negative*, and the finite axis AA' as likewise negative. The ellipse property, $SP+HP=AA'$, thus becomes, in the case of the hyperbola, $SP + (-HP) = (-AA')$. So in Art. 15, Cor., in the case in which the focal chord meets the nearer branch of a hyperbola in Q and its further branch in P, we must write $\dfrac{1}{(-SP)} + \dfrac{1}{SQ} = \dfrac{2}{L}$. In accordance with the same principle the normal to a hyperbola at P bisects the *interior* angle between PS and $P \infty H$; and if two tangents TO, TO' be drawn from T to the same branch of a hyperbola, the angle STO will be equal to the angle between TO' and $T \infty H$.

Combining the results of the two preceding paragraphs, we infer from the property $SY.HZ = CB^2$ in the ellipse, that $SY.(-HZ) = (-CB^2)$ in the hyperbola; and from $SP.HP = CD^2$ in the former, that $SP.(-HP) = (-CD^2)$ in the latter. On the same principle, Prop. I. assumes the form

$$PN^2 : AN.(-A'N) = (-CB^2) : CA^2$$

in the case of the hyperbola. In Art. 36, Cor., if the focal chord FF' be positive CD^2 will be negative, and *vice versa*. From the result $PF.CD = CA.CB$, obtained in evaluating the conjugate circumscribing parallelogram of the ellipse, we deduce in the case of the hyperbola that $(-PF).\sqrt{(-1)}\,CD = (-CA).\sqrt{(-1)}\,CB$; and the final result is independent of the factor $-\sqrt{(-1)}$.

THE BIFOCAL DEFINITION.*

PROPOSITION XVI.

48. *The tangent to a bifocal conic makes equal angles with the focal distances of its point of contact.*

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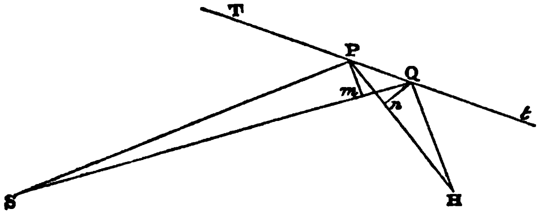
* The theorems in this section have for the most part been already proved in other ways; but they are here derived from the bifocal property, $SP \pm HP = AA'$,

(i) Let $TPQt$ be a chord which meets the conic in the adjacent points P and Q, and let SM be taken equal to SP upon SQ, and Hn equal to HQ upon HP, so that



$$Pn = HP \sim HQ = SQ \sim SP = Qm.$$

Now in the isosceles triangles SmP, HnQ, as the angles at S and H are diminished indefinitely, each of the remaining angles becomes very approximately a right angle; and hence ultimately, when Q coalesces with P, the triangles PmQ, PnQ become *right-angled* at m and n.

And since these triangles have the common hypothenuse PQ, and it has been shewn that Pn, Qm are equal, therefore the angle SQT of the one triangle is equal to the angle HPt of the other. That is to say, the point Q being supposed to have coalesced with P, the angle SPT is equal to HPt, or the tangent at P makes equal angles with the focal distances of that point.

(ii) Or, conversely, taking EUCLID's definition of a tangent, we may proceed as follows.

In the case of the ellipse, P being any point on the curve, and S, S' the foci, in SP produced take Ps equal to PS', draw the bisector of the angle $S'Ps$, and take any point Y upon it.

Then since $S'Y$ is evidently equal to sY,

$$SY + S'Y = SY + sY > Ss > SP + S'P;$$

that is to say, the sum of the focal distances of any point other than P upon the bisector of the angle SPs is greater than the

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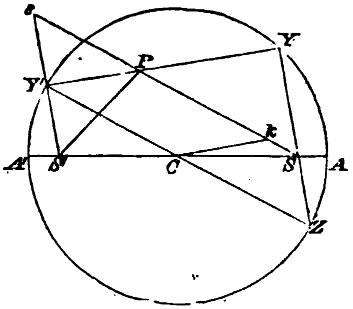
by which the ellipse and hyperbola are sometimes *defined*. From this property it is evident that an ellipse may be traced with the point of a pencil moved along a string, whereof the ends are fixed, so as to keep it stretched.



transverse axis; every such point therefore lies without the curve, and the bisector of the angle $S'Ps$ is the tangent at P. The proof applies *mutatis mutandis* to the hyperbola.

Corollary.

The normal at P bisects the angle between the focal distances of P, *estimated within the curve;* and hence it easily follows that G being the point in which it meets the axis

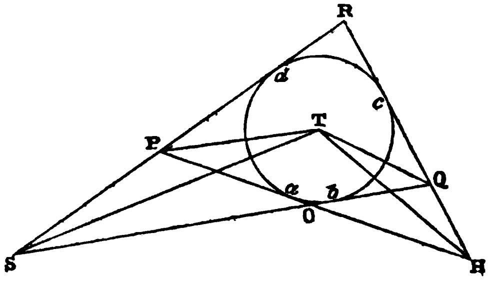
$$SG : SP = HG : HP = CS : CA.$$

PROPOSITION XVII.

49. *The two tangents to a bifocal conic from any external point subtend equal or supplementary angles at either focus.*

This may be deduced from the LEMMA that *a circle can be inscribed in any quadrilateral which is such that the sum or difference of two of its sides is equal to the sum or difference of its other two sides.*

If four straight lines touch a circle at points a, b, c, d, and if they form by their intersections the quadrilaterals $POQR$,



$SPHQ$, $SOHR$, as in the diagram, it readily follows, from the equality of the two tangents (as Oa, Ob) from any external point to a circle, that

$$OP + QR = OQ + PR;\quad SP + HP = SQ + HQ;$$

and $$SO - HO = SR - HR;$$

and, conversely, if one of these relations be given, a circle can be drawn touching the four straight lines.

The LEMMA being assumed, if P, Q be points on a conic whose foci are S and H, then since

$$SP \pm HP = SQ \pm HQ,$$

a circle can be described touching SP, SQ, HP, HQ, and having its centre T at the point of concourse of the external or internal bisectors of the angles SPH, SQH; that is to say, at the point of concourse of the tangents at P and Q.*

And since SP and SQ touch a circle whose centre is T, therefore ST bisects the angle between them; and in like manner HT bisects the angle between HP and HQ. That is to say, the tangents TP, TQ to the conic subtend equal (or supplementary) angles at either focus.

Corollary.

One or other of the angles between any two tangents to a conic is equal to half the sum or difference of the angles which their chord of contact subtends at the foci.† Taking the case of a pair of tangents TP, TQ to an ellipse whose chord of contact (as in the next figure) does not pass between the foci, we may shew, by equating the angles of the quadrilaterals $TPSQ$, $TPHQ$ to eight right angles, that the exterior angle between the tangents is an arithmetic mean to PSQ and PHQ.

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* If with S and H as foci an ellipse be drawn through P, Q and a hyperbola through O, R, their common diameter through T will bisect PQ in the one curve and OR in the other. Hence the middle points of the three diagonals of any quadrilateral in which a circle can be inscribed lie upon one diameter of the circle.

† This theorem, with its analogue for the parabola (Art. 28), was proved by BOSCOVICH (*Sectionum Conicarum Elementa*, § 184). In a slightly different form it will be noticed again in Scholium D.

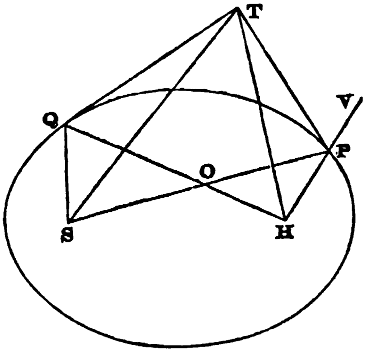
PROPOSITION XVIII.

50. *The two tangents to a bifocal conic from any external point are equally inclined to its two focal distances, each to each.*

Let TP, TQ be the tangents at P, Q to a conic whose foci are S and H; then will the angles STQ, HTP be *equal* to one another, unless P and Q lie on the same branch of a hyperbola, in which case they will be *supplementary*.

(i) In the case of the ellipse, produce HP to any point V, and let SP, HQ intersect in O.

Then since the tangent at P bisects the angle SPV, and HT bisects the angle PHQ, therefore



$$\angle HTP = TPV - THP = \tfrac{1}{2}SPV - \tfrac{1}{2}PHQ = \tfrac{1}{2}POH;$$

and in like manner it may be shewn that the angle STQ is equal to $\frac{1}{2}QOS$ or $\frac{1}{2}POH$.

Hence TP and TQ make equal angles with the focal distances of T.[*]

(ii) If the tangents be drawn to opposite branches of a hyperbola, they will still make equal angles with ST and HT.

(iii) If both tangents be drawn to the S-branch of a hyperbola, and if HT be *produced* to a point H', it will be

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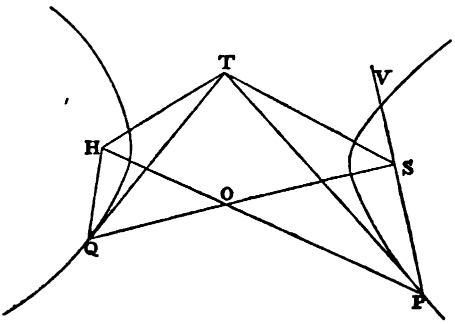
[*] In Art. 87 the points S, P, M, R and S', P', M', R' are concyclic, as will likewise be the case (Art. 9, Cor. 1) if RR' be *any straight line*, and P its polar. Hence, if X, X' be the feet of the directrices, $SPR = SMX = S'M'X' = SPR'$. It follows by the proposition that *the intercepts on any chord made by the curve and its directrices subtend equal* (*or supplementary*) *angles at the pole of the chord.*



seen that they make equal angles with ST and $H'T$ respectively, which is in accordance with the principles of Scholium C.

Corollary 1.

The bisectors of the angles between any two tangents to a bifocal conic are the tangent and normal to the two confocals through their point of concourse.

Corollary 2.

The triangle whereof the base is equal to the transverse axis of a conic and its remaining sides to the focal distances of any external point has its vertex angle equal to the angle between the tangents to the conic from that point and its remaining angles to the angles which either tangent subtends at the foci. For in the first figure, if SP produced to a point H' be equal to the transverse axis, it is evident that the triangle $TH'P$ is identically equal to THP, and hence that the triangle STH' is of the specified linear and and angular dimensions.

SCHOLIUM D.

The theorem of Cor. 2 may be expressed as follows. If the straight lines in either diagram be regarded as a framework jointed at their ends, and if S and H be drawn apart (or brought together) until the distance between them is equal to AA', then will SPH and SQH become straight lines, and the angles at the joints (except SPH and SQH) will be equal to the angles at the same points in the original figure, the inner and outer angles at T being *interchanged*. The angle STH in the deformed figure is in general equal to the angle between the tangents, but in the third case of the proposition it becomes *supplementary* thereto. The following are some applications of this theorem.

(i) If the angle between the tangents be *a right angle*, then $ST^2+HT^2=AA'^2$, whence the property of the director circle readily follows. In any case, the angle θ between the two tangents from any given point T satisfies the relation

$$ST^2+HT^2-2ST.HT\cos\theta=AA'^2.$$

(ii) Since the exterior vertex angle of the triangle is equal to the sum of its angles at S and H, therefore the supplement of the angle between any two tangents to an ellipse is equal to *the sum of the angles which either tangent subtends at the foci.* In the hyperbola, as may be proved in like manner, the *difference* of the angles which either tangent subtends at the foci will be equal or supplementary to the angle between the tangents, according as they are drawn to opposite branches of the curve or to the same branch. In the one case, P and Q in the deformed figure will lie on opposite sides of SH, and in the other case on the same side of it.

(iii) It may be shewn from the deformed figure that

$$TP^2:TQ^2=SP.HP:SQ.HQ.$$

Hence, having deduced (Art. 45, Cor. 2) from the bifocal definition that

$$SP.HP+CP^2=CA^2+CB^2=SQ.HQ+CQ^2,$$

we infer, taking the case in which CP, CQ are *conjugate*, and therefore equal and parallel to the tangents at Q and P, that

$$SP.HP=CQ^2;\quad SQ.HQ=CD^2;\quad CP^2+CQ^2=CA^2+CB^2;$$

and hence, that *in all cases* the tangents are as the parallel radii.

PROPOSITION XIX.

51. *At any point of a bifocal conic the projection of the normal terminated by the conjugate axis upon the distance of the point from either focus is equal to the semi-axis transverse.*

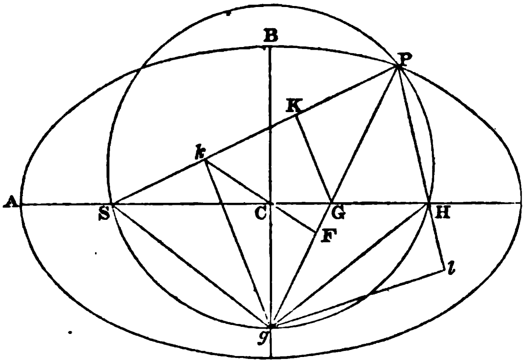
Let the normal at P meet the conjugate axis in g, and draw gk and gl perpendicular to SP and HP respectively. Then since the normal bisects the angle kPl, therefore

$$Pk=Pl,\text{ and } gk=gl.$$

And since also the hypothenuses of the right-angled triangles gkS, glH are equal, therefore the side Sk of the one is equal to the side Hl of the other.

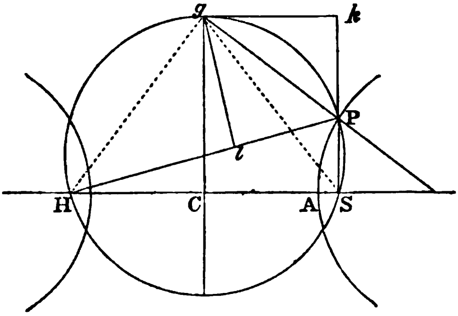
Hence, in the first figure,

$$SP - Pk = Pl - HP = Pk - HP;$$



and in the second figure,

$$SP + Pk = HP - Pl = HP - Pk.$$



Therefore $$Pk = \tfrac{1}{2}(SP \pm HP) = CA.$$

Corollary 1.

If GK be drawn, as in Art. 11, it follows from this proposition, together with Art. 52, Cor. 2, that $PK.CA = CB^2$.

Corollary 2.

By the converse of Art. 38, Cor. 3, the diameter Ck is parallel to the tangent at P, or perpendicular to the normal;

if therefore it meet the normal in F,

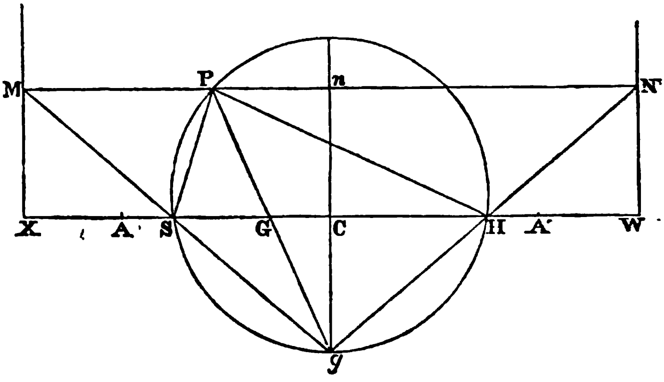
$$Pg.PF = Pk^2 = CA^2,$$

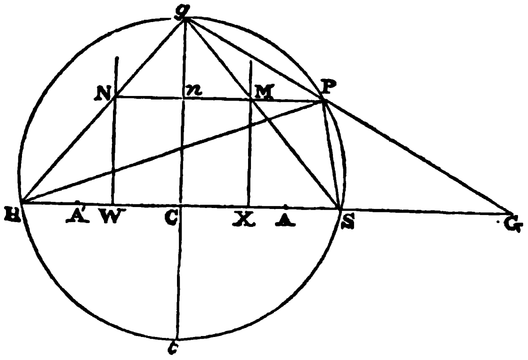
and $$PG.PF = PK.Pk = CB^2.$$

PROPOSITION XX.

52. *The distance of any point on a bifocal conic from either focus varies as its distance from a corresponding fixed directrix perpendicular to the axis.*

Bisect the angle between the focal distances of any point P of the locus, estimated *within* the curve, and let the bisecting line meet the axes in G and g. Let a parallel to the transverse axis through P meet gS in M and gH in N; and let perpendiculars be drawn to the axis through those points, meeting





it in X and W. We shall shew that these perpendiculars are *directrices*.

Since $\qquad \angle SMP = gSH = gHS = GPS,$

the points g, S, P, H being evidently concyclic; therefore, in the similar triangles SPM, SGP,

$$SP : PM = SG : SP = SG \underset{\sim}{+} GH : SP \underset{\sim}{+} PH = CS : CA;$$

and therefore SP is in *a constant ratio* to PM.

And since $\quad SH : MN = SG : PM = CS^2 : CA^2,$

therefore MN, or $2CX$, is constant, and MX is *a fixed straight line.*

The curve may therefore be described with S and MX, or, in like manner, with H and NW, as FOCUS and DIRECTRIX.

Corollary 1.

The line PG in the above construction being evidently the normal at P, it follows that *the focal distances of the point in which the normal at any point of the curve meets the conjugate axis pass through the feet of the perpendiculars from that point to the directrices.* Hence an obvious construction for the normals from g with the help of the ruler only.

Corollary 2.

Since $\qquad Gg : Pg = SG : PM = CS^2 : CA^2,$

therefore $\quad PG : Pg = CA^2 \sim CS^2 : CA^2 = CB^2 : CA^2.$

SCHOLIUM E.

The name FOCUS has reference to the *optical* property in relation to the conic of the points to which it is now commonly applied, viz. that rays proceeding from one of them and reflected at the curve would converge to or diverge from the other (Art. 37), or become parallel to the axis (Art. 25).

APOLLONIUS, who introduces the foci somewhat late in his treatise, proves their properties in the following order (Lib. III. props. 45—52). Starting with the property, $AS \,.\, A'S = CB^2$, he shews that the intercept on the tangent at any point P by the tangents at the vertices subtends right angles at the foci; that the tangent at P and either of the fixed tangents make equal angles with the focal vectors to their point of concourse (a special case of Art. 50); that a pair of the focal vectors to its points of concourse with the two fixed tangents intersect on the normal at its point of contact; that it makes equal angles with the focal distances of that point; that the axis subtends a right angle at the foot of the focal perpendicular SY upon the tangent; that the diameter parallel to HP meets the tangent upon the circumference of the circle on AA', viz. in the same point Y; and lastly, that $SP \mp HP = AA'$.

The property of the focus and directrix by which we defined the general conic is given in the Mathematical Collections of Pappus, Lib. vii. prop. 238 (vol. ii. p. 1012, *ed.* Hultsch, Berol., 1877).

Newton, to whom some later writers were indebted for their acquaintance with the property, mentions it in the *Principia*, at the end of Lib. i., Sect. iv., in connexion with his construction (a modification of De la Hire's Lib. viii. 25, to which he refers) for an orbit whereof a focus and three points are given, viz. by determining a point of the directrix upon the chord joining each pair of given points on the curve.

EXAMPLES.

201. If AB and CD be equal portions of two straight bars, and if they be connected by hinges with two equal bars AD, BC in such a manner that initially AB and CD form opposite sides of a rectangle and AD and BC its diagonals; then (1) if a *side* of the rectangle be fixed, the remaining parts of the framework being moved about in any way in a plane, the intersection of the cross bars will trace an ellipse; or (2) if a *diagonal* be fixed, the continuations of AB and CD will cross one another upon the arc of a hyperbola.

202. The sides AD, DC of a rectangle $ABCD$ are divided into the same number of equal parts, and straight lines are drawn from B and A respectively to the points of section. Shew that the corresponding lines of the two series meet on an ellipse whose axes are equal to the sides of the rectangle.

203. A parallelogram $ABCD$ has its diagonal AC at right angles to the side AB. If CD be divided into any number of equal parts and straight lines be drawn from A to the points of section, and if AC be divided into the same number of equal parts and straight lines be drawn from B to the points of section, then will the corresponding lines in the two series meet on a hyperbola.

204. The straight lines joining the vertices of a conic to opposite ends of any segment of the directrix which subtends a right angle at the focus intersect on the curve. Hence shew how to trace an ellipse or hyperbola, and prove that the ellipse is a closed curve giving wholly between the perpendiculars to the axis at its vertices.

205. The locus of the point at which the distances of either vertex from a focus and directrix subtend equal angles is the auxiliary circle. Hence shew that the ellipse cuts the ordinates of its auxiliary circle in a constant ratio.*

206. If AM and $A'M$ be taken on the axis of a conic equal to the focal distances of any point P on the curve, shew that

$$CP^2 = CB^2 + CM^2;$$

and deduce that the square of the ordinate of P varies as the rectangle contained by its abscisses.

207. If a circle be drawn through the vertices of a conic and any point on the curve, find the locus of the second point in which it meets the ordinate of the former.

208. If two ellipses whose major axes are equal have a common focus, they will intersect in two points only; and their common chord will be at right angles to the straight line joining their centres.

209. Given a chord of a parabola and the direction of its axis, the locus of the focus is a hyperbola whose foci are at the extremities of the chord.

210. The straight lines from either focus of a conic to the ends of a diameter make equal angles with the tangents thereat.

211. A circle can be drawn through the foci and the intersections of any tangent with the tangents at the vertices.

212. The intercept on any tangent by the tangents at the extremities of a focal chord subtends a right angle at the focus.

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* If p be a point on the locus, then $Sp : PX = SA : AX = SA' : A'X$, and therefore $A'p$ is the outer bisector of the angle SPX, and is at right angles to Ap. Hence $pN^2 - PN^2 = Sp^2 - SP^2 = e^2 (PX^2 - NX^2) = e^2 . PN^2$, and PN^2 varies as pN^2, or as $AN . A'N$.

213. The major axis is the maximum chord of an ellipse, and the minor axis is its least diameter. When is the angle subtended at the curve by the straight line joining the foci a maximum?

214. The focal vectors to any two points on a conic meet in two other points lying on a confocal conic, and the tangents at the two pairs of points cointersect. Examine the case in which the focal vectors are drawn to opposite extremities of a diameter.

215. Find the locus of the centre of a circle which touches two fixed circles. Also, if two circles, the sum or difference of whose radii is constant, be described about fixed points, find the locus of the centre of a circle of given magnitude which touches both of them.

216. If PQ be a chord of an ellipse, and if the ordinates of its extremities to either axis be produced to meet the corresponding auxiliary circle in p and q, then will PQ and pq meet on that axis; and if the tangent at p meet the same axis in T, then will TP touch the ellipse at P, and the abscissa of P will be a third proportional to CT and CA.

217. If any tangent to an ellipse meet the axes in T and t, the tangents from those points to the major and minor auxiliary circles respectively will be parallel two and two, and their four points of contact will lie on two diameters of the ellipse.

218. The perpendiculars to the axes from the points in which a common diameter meets the two auxiliary circles of an ellipse intersect two and two on the curve. Hence shew how to construct an ellipse with the help of two fixed concentric circles.

219. If two points on a straight line move along the arms of a right angle, any other point on the line will trace an ellipse whose semi-axes are equal to the segments of the line between that point and the former two.*

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* The moving line may be supposed parallel to the "common diameter" in Ex. 218. This theorem explains the construction of the *Elliptic Compasses*.

220. The ordinates of the points in which any tangent to an ellipse or hyperbola meets the curve the tangents at its vertices and the conjugate axis are proportionals, and the product of the extremes or the means is equal to the square of the semi-axis conjugate.

221. If any ordinate to either axis be bisected in O, and AO, $A'O$ meet the tangents at A' and A in T' and T respectively, then will TT' be the tangent at the extremity of the ordinate; and the straight line joining the intersections of ST, HT' and ST', HT will be the normal at that point.

222. If two ellipses with equal axes be placed vertex to vertex, and one of them then roll upon the other, each of its foci will describe a circle about a focus of the latter.

223. Given a central conic, shew how to find its centre, axes, foci, and directrices.

224. Given a focus and two points of an ellipse, the locus of the other focus will be a hyperbola. If instead of one of the two points the length of the axis be given, determine the loci of the centre and of the second focus.

225. Given one focus of a conic inscribed in a triangle, shew how to determine the other focus. If an ellipse inscribed in a triangle have one focus at the orthocentre, its other focus will be at the centre of the circumscribed circle, and its auxiliary circle will be the nine-point circle of the triangle. Examine the case in which a focus of the inscribed conic is at the centre of the inscribed circle of the triangle.

226. The angular points and the sides of a triangle being taken as the centres and directrices of three ellipses which have a common focus at the orthocentre, shew that the sum of the squares of their major axes is double of the sum of the squares of the sides of the triangle; their minor axes are equal to one another, the square of each being equal to the sum of the squares of the three latera recta; the sum of the squares of the eccentricities is equal to two; the intercepts made by the ellipses upon the sides of the triangle are conjugate diameters;

the perpendiculars of the triangle are the common chords of the ellipses, and their six poles lie on three focal chords parallel to the sides of the triangle.

227. If CP be any radius of a conic, and if a parallel to it be drawn from the vertex A to meet the curve in Q and the conjugate axis in R, shew that $AQ.AR$ is equal to $2CP^2$.

228. A straight line equal to the radius of a circle slides with one end on a fixed diameter and the other end on the convexity of the circumference. Shew that any intermediate point on the line traces an arc of an ellipse.

229. From a fixed point O a straight line OP is drawn to a given circle. Find the envelope of a straight line drawn through P at a constant inclination to OP.

230. The straight line joining the foci of a conic subtends at the pole of any chord an angle equal to half the sum or difference of the angles which it subtends at the extremities of the chord.

231. If CR be the projection of any radius CD upon the axis of the conic, and OL the ordinate of the middle point O of any chord[*] parallel to CD, prove that

$$OL.DR : CL.CR = CB^2 : CA^2.$$

Hence shew that the diameters of conics are straight lines, and obtain the relation between the inclinations of any two conjugate diameters to the axis.

232. The ellipse has a pair of equal conjugate diameters, which coincide in direction with the diagonals of the rectangle formed by the tangents at the ends of its axes, and which are equal to the sides of a square whose diagonals are equal to those of the said rectangle.

What is the corresponding property of the hyperbola?[†]

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[*] Art. 88 gives $PN^2 \sim QM^2$ in terms of $CN^2 \sim CM^2$; whence the required result readily follows.

[†] It has two pairs of (infinite) conjugate diameters which are in a ratio of equality, and each *pair* coincide in direction with one of the equi-conjugate diameters of the ellipse which has the same axes.

233. The common diameters of equal similar and concentric ellipses are at right angles to one another.

234. Find the loci of the centres of the four circles which touch the axis of a conic and the two focal vectors to any point on the curve.

235. A chord of a conic which subtends a right angle at the vertex passes through a fixed point on the axis.

236. If a hyperbola touch the sides of a quadrilateral inscribed in a circle, and if one focus lie on the circle, the other will also lie on the circle.

237. If three circles be described on the transverse axis and the two focal distances of any point of a conic as diameters, determine their radical centre.

238. If the tangent at a point P whose ordinate to either axis is PN meet the corresponding auxiliary circle in Y and Z, shew that C, N, Y, Z lie on a circle, and that PN bisects the angle YCZ.

239. If an ellipse and a hyperbola have the same axes, the director circle of the one will pass through the foci of the other.

240. The diagonals of any rectangle circumscribing a conic are conjugate diameters.*

241. The diagonals of any parallelogram circumscribing a conic are conjugate diameters, and the sides of any inscribed parallelogram are parallel to conjugate diameters.

242. The sum or difference of the reciprocals of any two focal chords at right angles, or of the squares of any two diameters at right angles, is constant.

243. The locus of the centre of an ellipse which slides between two straight lines at right angles is a circle.

244. The circle described upon the straight line joining the foci of a conic meets the conjugate axis in two points such that

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* This appears from De la Hire's original proof of the property of the director circle, in which he assumed the theorems of Arts. 39, 45.

the sum of the squares of the perpendiculars therefrom to any tangent is equal to half the square upon the transverse axis.

245. Determine the positions of a chord of an ellipse which subtends a right angle at each of the foci; and also the locus of the pole of a chord which subtends supplementary angles at the foci.

246. The opposite sides of a quadrilateral described about an ellipse subtend supplementary angles at either focus.

247. The angle which a diameter of an ellipse subtends at an extremity of the major axis is supplementary to that which its conjugate subtends at an extremity of the minor axis.

248. If a focal chord of a conic be drawn to meet at a given angle any tangent, or any chord subtending a constant angle at one of the foci, the locus of the point of intersection will be a circle.

249. To what does the theorem that confocal conics intersect at right angles reduce when the two foci coalesce?

250. The circle described about any point on the axis of a hyperbola so as to cut its auxiliary circle orthogonally meets the ordinate through that point upon the circumference of an equilateral hyperbola.

251. The pole of any straight line with respect to a central conic may be found by joining the points in which it meets the directrices to the nearer foci, and drawing perpendiculars through the latter to the joining lines.

252. The straight lines joining any point to the intersections of its polar with the directrices touch a confocal conic.

253. In Art. 4, if the centre of the circle be taken midway between the vertices of the conic, shew that the directrix will be the polar of the focus with respect to the circle. Hence shew that every chord of an ellipse or hyperbola which passes through its centre is bisected at that point, and that the curve is consequently symmetrical with respect to its conjugate axis.

254. The locus of the vertex of a triangle whose base and the ratio of whose sides are given is a circle, whereof a diameter is determined by dividing the base of the triangle externally and internally in the given ratio. Hence shew that a straight line parallel to the axis of a conic meets the curve in general in two points, and that the two points are equidistant from a fixed straight line parallel to the directrix.*

255. Apply the same method to determine the points in which any assumed straight line meets the conic;† and also to shew that the diameters of conics are straight lines.

256. Prove the following construction for drawing tangents to a conic whose foci are S and H from a given external point O. About O with radius OS describe a circle, and about H with radius equal to the transverse axis describe a circle; then will the required points of contact lie upon the straight lines drawn from H to the points in which the two circles intersect. Prove also that the two tangents as thus determined subtend equal or supplementary angles at either focus.

257. If the diagonals of a quadrilateral circumscribing an ellipse meet at its centre the quadrilateral must be a parallelogram.

258. If a principal ordinate meet an ellipse in P and its auxiliary circle in Q, the distance of the former point from either focus will be equal to the perpendicular from that focus to the tangent at Q.

259. The locus of the middle point of a focal chord of a conic is a similar conic. In what other cases will the locus of the middle point of a chord of a conic be a similar conic?

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* The two points are determined as follows. Let the parallel meet the directrix in Q, and let Z and Z' divide SQ in a ratio equal to the eccentricity. Describe the circle on ZZ', and let it cut the parallel in P and P'. The projection of the centre of this circle upon the axis of the conic evidently lies midway between the projections of Z and Z' upon the same, that is to say, midway between the vertices of the conic.

† If the assumed line meet the directrix in Q and make an angle α with the axis, divide SQ in the ratio $e \cos \alpha$, and upon the intercept between the two points thus determined describe a circle cutting the assumed line in P and P', which will be the points required.

260. If tangents be drawn to a conic at the extremities of a pair of conjugate radii, the focal vectors to their point of concourse will meet those diameters in four points lying on a circle.

261. If SY be a focal perpendicular upon the tangent at P and CD the radius parallel to the tangent,

$$SY^2 : CB^2 = SP^2 : CD^2 = SP : 2CA \pm SP.$$

262. The feet of the perpendiculars from one focus of a conic to a pair of tangents are on a straight line at right angles to the distance of their point of intersection from the other focus.

263. Find the greatest or least value of the sum of the squares of the focal perpendiculars on any tangent.

264. The normal at any point of an ellipse is a harmonic mean to the focal perpendiculars upon the tangent at that point.

265. If one focus of a conic which touches the sides of a triangle be at its centroid, the distances of the other focus from its sides will be as the lengths of those sides.

266. If the normal at P meet either axis in G, shew that any circle through those points will intercept on the focal distances of P chords whose sum or difference has one or other of two constant values.

267. The diameters parallel to the tangent and normal at P intercept on SP a length equal to HP; and the latter diameter meets SP on the circumference of a circle.

268. The circle described upon the central abscissa of the foot of the normal at any point is cut orthogonally by a circle described about that point and equal to the minor auxiliary circle.

269. The intercepts on the focal vectors to the points of contact of a conic with any circle which touches it in two points have one or other of two constant values.

270. Any tangent and its normal meet either axis in points T and G such that $CG.CT = CS^2$. In what cases does the normal meet the conjugate axis without within or upon the curve? If a circle touch an ellipse in two points and also touch its directrices, its centre will be at an end of the minor axis.

271. The intercept made by the directrices upon any normal chord of a conic subtends at the pole of the chord an angle equal to half the sum or difference of the angles which the distance between the foci subtends at the extremities of the chord.*

272. If two chords be drawn from any point of a conic equally inclined to the normal at that point, the tangents at their further extremities will intersect upon the normal.

273. Supplemental chords of a conic which are equally inclined to the curve at their common point have their poles upon the director circle, and their sum or difference is equal to the diameter of the same.†

274. The normals at the extremities of any two conjugate radii meet on the diameter which is at right angles to the chord joining those points; and any two normals at right angles to one another intersect on the diameter which bisects the chord joining the points at which the normals are drawn.

275. If on the normal at P a length PQ be taken equal to the semi-diameter conjugate to CP, the locus of Q will be a circle of radius $CA \pm CB$.

276. If normals be drawn to an ellipse at the ends of any chord parallel to one of its equal conjugate diameters, the locus of their intersection will be a line perpendicular to the conjugate diameter.

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[*] If the normal at P meet the S-directrix in R, and O be the pole of the normal, the circle on OR will pass through S and P.

[†] See Wolstenholme's *Book of Mathematical Problems*, No. 493 (London and Cambridge, 1867). If PQ, PQ' be two such chords they will evidently touch a confocal, and the parallel chords will also touch the same. Hence the tangent at Q will make equal angles with PQ and a parallel to PQ', and will be at right angles to the tangent at P. Otherwise thus: the tangent and normal at P divide QQ' harmonically, and the normal is an ordinate of QQ' and parallel to the tangent at Q.

277. With the construction of Art. 52, shew that

$$Sg : Pg = PG : SM = Pg : gM = CS : CA.$$

Shew also that the loci of the middle points of PG and Pg are conics.

278. If N be the projection of any point P on a conic upon the tangent at a given point O, find the relation between the lengths of ON and PN.*

279. If PQ be a chord of a conic which subtends a right angle at a given point O on the curve, and MN be the projection of the chord upon the tangent at O, shew that

$$\frac{PM}{OP^2} \pm \frac{QN}{OQ^2} = \text{a constant},$$

and that PQ passes through a fixed point on the normal at O.†

280. If PG be the normal at any point P on an ellipse, Q the point in which the ordinate of P meets the auxiliary circle, and R the point in which CQ meets the ellipse, then will QG be parallel to the normal at R. Moreover, if any two chords of an ellipse be at right angles, its diameters conjugate to the corresponding chords of its auxiliary circle will be conjugate diameters of a certain concentric ellipse.

281. If N be the projection of any point in the plane of a conic upon its transverse or conjugate axis, and T the point in which its polar meets the same, shew that $CN.CT$ is equal to CA^2 in the former case and to CB^2 in the latter.‡

282. If P, P' be any two points whereof the one lies on the polar of the other, and N, N' be their projections on the transverse axis, then

$$\frac{CN.CN'}{CA^2} \pm \frac{PN.P'N'}{CB^2} = 1.$$

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* Let the normal at O meet the curve again in H, and let NP meet the tangent at H in M; then ON varies as a mean proportional to PN and PM (Ex. 73, and note p. 65), and also as $PN + PM \sim OH$.

† The intercept on the normal varies as the diameter conjugate to CO.

‡ Let the diameter through the assumed point O meet the directrix in V, and let its polar meet the directrix in R, which will be the orthocentre (Art. 14) of the triangle STV. Then, TV being parallel to SO (Art. 17), $CN : CX = CO : CV = CS : CT$.

283. If P and Q be any two points on a conic, M and N their projections on either axis, and T the point in which PQ meets the same, shew that

$$CT(PM - QN) = PM.CN - QN.CM.[*]$$

284. Shew also that if CL be the central abscissa upon either axis of the point of concourse of the normals at P and Q,[†] then $CL.CT$ varies as $CM.CN$.

285. If the normals at four points of a conic cointersect, and if the three pairs of chords joining those points meet either axis in T, T'; U, U'; V, V'; shew that

$$CT.CT' = CU.CU' = CV.CV'.$$

286. If the normals at four points of an ellipse or hyperbola cointersect, each pair of chords joining the four points will be parallel to a pair of conjugate diameters of the hyperbola or ellipse which has the same axes, and will meet either axis in points which are conjugates in an involution determined by the latter curve.[‡]

287. The sum or difference of the squares of the perpendiculars from the extremities P and D of any two conjugate semi-diameters of a conic upon a fixed diameter of the same is constant; and if CN and CR be the abscissæ of P and D upon that diameter, and $2CP'$ be its length and $2CD'$ the length of its conjugate, then

$$CN^2 \pm CR^2 = CP'^2;\quad PN^2 \pm DR^2 = CD'^2;$$

and $$PN : CR = DR : CN = CD' : CP'.$$

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[*] Equate the areas $(CPT - CQT)$ and $(CPM + PMNQ - QCN)$.

[†] If L be the projection upon either axis of any point O on the normal at P, and G be the point in which the normal meets that axis, then $OL : PM = CG \sim CL : MG$; and CG and MG vary as CM.

[‡] The pairs of points in Ex. 285 determine an involution whose centre is C, and to which the ends of the axis in question likewise belong. If these points be on the transverse axis, $CT.CT' = \&c. = CA.CA' = -CA^2$; and if on the conjugate axis, $CT.CT'$ is equal to $-CB^2$ in the case of the ellipse, and to $+CB^2$ in the case of the hyperbola.

288. The vertices of the conjugate circumscribing parallelogram of a conic lie on a similar conic, and their polars envelope another similar conic.

289. The inscribed parallelogram whose diagonals are at right angles envelopes a circle, the reciprocal of the square of whose radius is equal to $\frac{1}{CA^2} \pm \frac{1}{CB^2}$.

290. If the polar of any point on an ellipse with respect to its minor auxiliary circle meet the major and minor axes in H and K, shew that

$$\frac{CB^2}{CH^2} + \frac{CA^2}{CK^2} = \frac{CA^2}{CB^2}.$$

291. Supplemental chords being drawn to a conic from its vertices, the perpendiculars to them at their common point make an intercept equal to the latus rectum upon the axis.

292. If an ordinate of any diameter meet the curve in P, the diameter in M, and any two supplemental chords drawn from its extremities in Q and R, shew that PM is a mean proportional to QM and RM, and that QR is bisected by the tangent at the intersection of the chords.

293. If two conjugate semi-diameters CP, CD, or their prolongations, make an intercept $P'D'$ upon a line which is parallel to PD and meets the conic in Q, shew that

$$QP'^2 \pm QD'^2 = PD^2.$$

294. From extremities of two conjugate diameters of an ellipse a pair of parallels are drawn to any tangent; if any diameter meet these parallels in P and Q and the tangent in R, shew that

$$CP^2 + CQ^2 = CR^2.$$

295. If PP' and DD' be conjugate diameters of a hyperbola and Q any point on the curve, then will $QP^2 + QP'^2$ exceed $QD^2 + QD'^2$ by a constant quantity.

296. Find the axes of a given conic by means of Art. 44, Cor. 3.

297. Given two conjugate diameters of an ellipse, shew that the locus of the centre of the circle through their common point and the points in which a tangent parallel to one of them meets any two conjugate diameters is a straight line perpendicular to the other.

298. Given two conjugate semi-diameters CP, CD of an ellipse or hyperbola, prove the following construction for its axes. In the direction CP or in the opposite direction take PQ a third proportional to CP and CD; draw the tangent at P, and through C, Q draw a circle having its centre upon it; join C with the points in which the circle cuts the tangent. The directions of the axes being thus determined, shew how to find their extremities.

299. Given two conjugate diameters CP, CD of an ellipse, with centre C and radius CP describe a circle, and let KK' be its diameter at right angles to CP; then will the axes of the ellipse be equal to $KD \pm K'D$, and parallel to the bisectors of the angle KDK'.*

300. Shew also that, if DN be taken equal to CP and be placed so as to cut it at right angles in a point L, and if AB be that diameter of the circle round CLN which passes through D, then will CA, CB be the directions and DB, DA the lengths of the semi-axes of the ellipse.

301. Any point of or in the same straight line with a rod which slides between two fixed straight lines describes an arc of an ellipse.†

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* See the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. III. pp. 151, 227 (1866); and the *Messenger of Mathematics* (New Series), vol. v. p. 122 (1876).

† Of two conjugate radii of an ellipse let CP be the shorter and CD the longer; draw a perpendicular DE to CP, and in the prolongation of or within ED take DQ equal to CP. Then if a straight line KM equal to EQ slides between CP and CQ, the point which divides it into segments KO and OM equal to CP and DE will be a point on the curve, viz. one whose abscissa on CD is terminated by the perpendicular from M to CP. As in the special case of Ex. 219, $KO \, . \, OM = CA \, . \, CB$. See Leslie's *Geometrical Analysis, and Geometry of Curve Lines*, p. 257 (Edinburgh, 1821).

302. The chords joining the extremities of two diameters of a conic and of their conjugates respectively are either parallel or conjugate in direction. If a series of chords pass through a fixed point, the chords of the corresponding conjugate arcs have the same property; and the diameters through the two fixed points are at right angles.

303. With the orthocentre of a triangle as centre two ellipses are described, the one touching its sides and the other passing through its angular points; prove that these ellipses are similar, and that their homologous axes are at right angles.

304. The perpendiculars from opposite foci of a conic upon two conjugate diameters intersect on a concentric conic passing through the foci.

305. If a chord AP drawn from the vertex A be divided in such a manner that $AQ : PQ = CA^2 : CB^2$, shew that the perpendicular from Q to the line joining Q to the foot of the ordinate of P divides the transverse axis in the same ratio.

306. From the foot of the ordinate of any point P on a conic a parallel is drawn to AP to meet the diameter through P in Q; shew that AQ is parallel to the tangent at P. Shew also that the bisectors of the angles ASP and AHP intersect on the tangent at P.

307. If two conics whose transverse axes are equal be inscribed in the same parallelogram, their foci will be at the corners of an equiangular parallelogram.

308. Any one of a series of conterminous circular arcs may be trisected by drawing a pair of hyperbolas whose determining ratio is equal to two, and whose centres and vertices trisect the chord of the arc. How does it appear from this construction that the problem, to trisect a given angle, admits of three solutions?

309. If any two conics have a common focus, one pair of their common chords cointersect with the corresponding directrices, and the other pairs subtend equal or supplementary angles at that focus.

310. A diameter meets the conic in P, its auxiliary circle in Q, and the tangent at either vertex in T. Prove that when the diameter through T coalesces with the axis PT and QT are in the duplicate ratio of the axes.

311. The perpendicular drawn through any point of a conic to one of its focal distances and terminated by the conjugate diameter varies inversely as the principal ordinate of the said point, and the perpendiculars from the vertices upon the tangent at any point meet its focal distances upon fixed circles.

312. A parabola of given linear dimensions being drawn to touch any two conjugate diameters of a conic symmetrically, find the locus of its focus.

313. The tangent at P meets any two conjugate diameters in T, t, and TS, tH meet in Q; prove that the triangles SPT, HPt, TQt are similar, and also that the area of the triangle CPT varies inversely as CPt.

314. The two points of a conic at which a given chord subtends the greatest and least angles are at the extremities of a diameter equal to that which bisects the chord.*

315. If an ellipse touch a given ellipse at adjacent extremities A, B of its axes and also pass through its centre, the tangent at the latter point will be parallel to AB.

316. With the normal and tangent at any point of a conic as axes a conic is described touching an axis of the former at its middle point; shew that the foci of the conic so drawn lie on fixed circles, whose diameters are equal to the sum and difference of the axes of the given conic.

317. Two fixed points being taken in given parallel lines, a straight line revolves about each point and meets the opposite parallel. If the envelope of the line joining the points of

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* The chord must subtend *equal* angles at either point and a consecutive point on the curve. The two points therefore lie on segments of circles described upon the chord so as to touch the conic.

concourse be a conic touching the parallels at the fixed points, determine the locus of the point in which the revolving lines intersect.

318. A chord of a circle which subtends a right angle at a fixed point envelopes a conic whose foci are the fixed point and the centre of the circle.

319. A straight line being drawn through a fixed point S to meet a given pair of parallels in Y and Z, shew that the envelope of the circle on YZ as diameter is a conic, of which the parallels are directrices and S is a focus.

320. On the axis of a hyperbola whose determining ratio is equal to two a point D is taken at a distance from the focus S equal to the distance of S from the further vertex A', and $A'P$ is drawn through any point P on the curve to meet the latus rectum in K. Prove that DK and SP intersect on a certain fixed circle.

321. The parallelograms whose diagonals are any two diameters of a conic and their conjugates respectively are of equal area.

322. If tangents TP and TQ be drawn to an ellipse whose foci are S and H, and CP' and CQ' be the parallel semi-diameters,

$$TP.TQ + CP'.CQ' = TS.TH.^*$$

323. Find the locus of a point such that the tangents therefrom to a central conic contain with the semi-diameters to their points of contact an area of constant magnitude; and the locus of a point such that the product of its focal distances varies as the product of the tangents.

324. The distance between any point and any point on its polar is cut harmonically by the tangents at the extremities of any chord through either point.†

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[*] We have to shew that the triangle STH' (Art. 50, Cor. 2) is equal to $PTQ + P'CQ'$; which follows from Ex. 321, taking into account that $PCQ = \frac{1}{2}(PSQ + PHQ)$.

[†] In the tractate, *De Linearum Geometricarum Proprietatibus Generalibus*, forming the Appendix to *A Treatise on Algebra, &c.*, by COLIN MACLAURIN, late Professor of

325. If O and P be any two points in the plane of a conic whose centre is C, the perpendiculars from O and C to the polar of P are to one another as the perpendiculars from P and C to the polar of O.

326. If two conics be concentric and similarly situated, the pole of any tangent to the one with respect to the other will trace a concentric conic; and if the two conics be also similar the third will be similar to both. If the axes of the two be identical, the pole of any tangent to the one with respect to the other will lie on the former.

327. Find a point which has the same polar, and a line which has the same pole, with respect to three conics, whereof one has double contact with the other two.

328. An ellipse has double contact with each of two circles, whereof one lies within the other. Shew that its chords of contact with them meet in a fixed point on the line joining their centres; the locus of its centre is a circle passing through their centres; its eccentricity is constant; and the locus of its foci is a circle concentric with the outer given circle.

329. Any diameter of an ellipse varies inversely as the perpendicular focal chord of its auxiliary circle.

330. If a parallelogram circumscribing a conic have two of its angular points on the directrices, the other two will lie on the auxiliary circle.

331. If two parallelograms be constructed, the one by joining the ends of two parallel focal chords of a conic, and the other by drawing tangents to it at those points; the area of the

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Mathematics in the University of Edinburgh (LONDON, 1779), it is shewn (Sect. I. §§ 9–11. Cf. Salmon's *Higher Plane Curves*, Art. 60), that if a straight line revolving about a fixed point P meet a curve of the n^{th} order in n points, and the tangents at those points meet any assumed straight line through P in K, L, M, &c., then will $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}+$ &c. the constant; and if the assumed line through P meet the curve, viz. in the n points A, B, C, &c., this constant will be equal to $\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}$ + &c. In the particular case of Ex. 324, it is evident that the point P, its polar, and the tangents at the extremities of any chord through P divide any straight line from P harmonically.

one will vary directly, and that of the other inversely, as the projection of one of the focal chords upon the conjugate axis.

332. If SY, HZ be the focal perpendiculars upon the tangent at P to an ellipse, and SY', HZ' perpendiculars upon the tangents from P to a confocal ellipse, then will the rectangle $YY'.ZZ'$ be equal to the difference of the squares of their major (or minor) semi-axes.

333. Determine the condition that the intercept on any tangent to a bifocal conic by two fixed tangents may subtend equal angles at the foci.

334. The tangents to a conic from any point on a circle through its foci meet the circle again in two points such that the second tangents therefrom intersect upon the circle.

335. Given an ellipse and a circle through its foci; prove that their common tangents touch the circle in points lying upon the tangent to the ellipse at an extremity of its conjugate axis.

336. If the tangent to a conic at a given point be met by any two parallel tangents, the focal distances of the points of concourse will meet on a fixed circle, whose centre will be on the normal at the given point.

337. The product of the tangents to a conic from any point is to the product of its focal distances as the distances of the point from the centres of the chord of contact and of the conic respectively. If the tangents from any point to a conic be in a constant ratio to the parallel diameters, determine the locus of the point.

338. Given an ellipse and one of its "cercles directeurs,"* shew that an infinity of triangles can be described about the one and inscribed in the other, and that all will have the same orthocentre.

339. An ellipse may be described by means of an endless string passing round two fixed points. If one focus be taken

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* See the second paragraph of Scholium B (p. 90) and compare Ex. 225.

anywhere on a given straight line whilst the other remains constantly fixed, the envelope of all the ellipses described with a given string will consist of two arcs of parabolas.

340. Shew that the line joining any point outside a conic to its centre, and the radical axis of a pair of circles through the point, the one passing through the points of contact of the tangents from the point, and the other passing through the foci, are equally inclined to the focal distances of the point.

341. If a normal to a conic meet the curve again in Q and the directrices in R, R'; and if O be the pole of the chord and S, S' the foci; prove that SR, OR' and $S'R'$, OR intersect on the normal at Q. In the case of the parabola, any normal chord produced to meet the directrix subtends a right angle at the pole of the chord; and the polar of the middle point of the chord meets the focal vector to its point of concourse with the directrix upon the normal at its further extremity.

342. At any point P on the auxiliary circle of an ellipse a tangent is drawn meeting the axis in T, and PA, PA' are drawn to the vertices meeting the ellipse again in D and E; prove that the chord DE passes through T.

343. The polar of any point O with respect to a conic and the perpendicular to it from O meet either axis in points T and G such that

$$CG.CT = CS^2.*$$

344. If a point be taken anywhere on a fixed perpendicular to either axis of a conic, the perpendicular from it to its polar will pass through a fixed point on that axis.

345. If perpendiculars SY, HZ, CM, PN be drawn to the polar of any point P, and if PN meet the axis in G, shew that $SY.HZ = CM.NG$; $CM.PG = CB^2$; and that the normal at a point on the curve which has the same central abscissa as P is a mean proportional to NG and PG.

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* Comparing Ex. 281 (note), $CG : CS = CO : CV = CS : CT$.

346. The poles of a given straight line with respect to a series of confocal conics lie upon a second straight line perpendicular to the former. Hence shew that if a chord of one conic touch another conic having the same foci, the tangents at its extremities will meet on the normal at its point of contact, and conversely the foot of the perpendicular from their intersection to the chord will be its point of contact.

347. If a triangle inscribed in a conic envelope a confocal, its points of contact will lie severally on three of the four circles which touch the sides of the triangle.

348. If a chord of an ellipse be drawn touching a confocal ellipse, the tangents at its extremities meet the diameter parallel to the chord on the circumference of a fixed circle, and the intercept on the chord by the diameter parallel to either tangent is of constant length;* the chord varies as the parallel focal chord of the outer ellipse, and conversely a chord which so varies envelopes a confocal; the projection upon the chord of the normal (terminated by either axis) at an extremity thereof is of constant length; if any circle touch a given ellipse in two points, the chords which can be drawn to the circle from either point of contact so as to touch a fixed confocal are of constant length, and conversely the envelope of a chord of constant length drawn to the circle from either point of contact is a confocal ellipse. Examine the case in which the minor axis of the inner ellipse is evanescent.

349. If the arms of a right angle envelope two confocal ellipses the line joining the points of contact will envelope a third ellipse confocal with the former two; and if two parallel positions of each arm be taken, the perimeter of the parallelogram formed by joining the points of contact will be constant,

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* If a, b and a', b' be the semi-axes of the outer and inner ellipses and $\lambda^2 = a^2 - a'^2 = b^2 - b'^2$, the intercept on the chord is $\frac{ab}{\lambda}$; the projections upon it of the normals are $\frac{\lambda b}{a}$ and $\frac{\lambda a}{b}$; and its length in terms of the parallel focal chord f is $\frac{\lambda f}{b}$. See the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. IV. pp. 11–22 (1868), in which article several of the examples following are likewise solved.

and each pair of its adjacent sides will make equal angles with the tangent at their point of concourse.

350. An ellipse which has double contact with each of two fixed confocals has a fixed director circle; and an ellipse which has double contact with one of two fixed confocal ellipses, and has its foci at the ends of any diameter of the other, has a fixed auxiliary circle.

351. Four tangents being drawn to a conic, if one pair of their points of intersection lie on a confocal each of the remaining two pairs will lie on a confocal. If TP and TQ be a pair of tangents to a conic, and tangents be drawn from P and Q to a confocal and intersect in points S and H, then TP and TQ subtend equal or supplementary angles at S and H; the four tangents from P and Q touch one and the same circle; and

$$SP \pm HP = SQ \pm HQ.$$[*]

352. Given two confocal ellipses, shew that the latus rectum of any ellipse which has its foci on the inner fixed ellipse and touches the outer is of constant length.[†]

353. The locus of the centre of a conic which has four-point contact with a given conic at a given point is a straight line through the centre of the fixed conic.

354. Prove Graves' Theorem, that the sum of the tangents from any point on an ellipse to a fixed confocal ellipse exceeds the intercepted arc of the latter by a constant quantity.[‡] Prove also that the difference of the tangents to an ellipse from any point on a confocal hyperbola is equal to the difference of the segments into which the intercepted arc of the ellipse is divided by the hyperbola.

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[*] See the article referred to in the note on Ex. 348.

[†] With the notation used above, its length is $\frac{2\lambda^2}{ab}$.

[‡] See Salmon's *Conic Sections*, Art. 399. The theorem may also be deduced from Ex. 351 as in the article referred to in the note. Adding the perimeter of the inner ellipse, we see that the outer confocal may be described with the help of a loop of string placed round the inner curve, a construction which becomes equivalent to that of the note on Art. 48 when the inner ellipse reduces (by the evanescence of its minor axis) to the line joining its foci.

355. If three ellipses be described as in Ex. 352, having their six foci at three points on the inner confocal, the sum of the areas of their minor auxiliary circles will be constant. Moreover, if any number of ellipses be described with the same number of points on a given ellipse as foci (every such point being a focus of two of them), and if the several ellipses touch as many fixed confocals, the areas of their minor auxiliary circles will be connected by a linear relation.*

356. If every vertex but one of a polygon circumscribing a conic trace a confocal conic, its remaining vertex will likewise trace a confocal, and the perimeter of the polygon will be constant.

357. If two points trace an ellipse (in the same direction) with velocities which are always as the focal chords parallel to the tangents at those points, the tangents will intersect on a fixed confocal ellipse, and their angular velocities about their points of contact will be as the central perpendiculars upon them.

358. If a parallelogram can be inscribed in an ellipse whose semi-axes are A and B so as to envelope a coaxal ellipse whose semi-axes are a and b,

$$A^2 - a^2 : a^2 = b^2 : B^2 - b^2.$$

359. If a single quadrilateral can be described about one of two given conics and inscribed in the other, any number of quadrilaterals can be so described, and they will have one diagonal in common.†

360. If the normal at any point P to an ellipse meet the two perpendicular tangents to a confocal ellipse in K and L, shew that $PK.PL$ is constant and equal to the difference of the squares of their major or minor semi-axes.

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* If β be the minor semi-axis of one of the variable ellipses and ϕ the arc joining its foci, then $\beta^2 = c\,(\phi + c')$, where c and c' are certain constants (Exx. 352, 354).

† The quadrilateral and its circumscribing conic can be projected into a rectangle and a circle, which latter must be the director circle of the projection of the inscribed conic. See also Ex. 382.

361. If from the intersections of any two parallel tangents to an ellipse with the tangent at a fixed point four tangents be drawn to a given confocal ellipse, the four intersections of the latter will lie on a certain circle having its centre on the normal at the fixed point; and the radius of the circle and the intercept made by its centre upon the normal will vary as the perpendicular diameter of the outer conic.*

362. If a tangent to a conic (or other curve) cuts off a constant area from another, it will be bisected at its point of contact, and conversely.†

363. A central conic which passes through four given points has a pair of conjugate diameters parallel to the axes of the two parabolas which can be drawn through the same four points.‡

364. Give a construction for finding a point P such that if straight lines PQ, PR, PS, PT be drawn from it to meet four given straight lines AB, CD, AC, BD at given angles, the rectangle $PQ.PR$ may be in a specified ratio to $PS.PT$. Hence shew how to draw the tangent at any given point on the locus of P; and determine a pair of conjugate diameters of the same.§

365. If a parallelogram $ASPQ$ has its opposite vertices A and P on a conic, and its sides AQ, AS meet the curve in

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* The four tangents in any assumed position will intersect on a circle (Art. 50). Any other quadrilateral inscribed in the same circle so as to envelope the inner ellipse will have the intersections of its opposite sides at points P and Q *on the fixed tangent* (Ex. 359); and it may be shewn conversely that the second tangents from P and Q to the outer ellipse are parallel. Making one of the parallel tangents coincide with the tangent at the fixed point, we see that the centre O of the circle must lie on the normal. Let M now be the intersection of the diagonals of the quadrilateral, N the fixed point, CD the semi-diameter conjugate to CN, and ρ the radius of the circle; then (Ex. 348) $NO = \frac{ab}{\lambda^2} CD$; $MN = \frac{\lambda^2}{ab} CD$; $\rho^2 = \left(\frac{a^2b^2}{\lambda^4} - 1\right) CD^2$. See also *Mathematical Questions, &c. from the* EDUCATIONAL TIMES, vol. XIII. p. 81.

† See SALMON'S *Conic Sections*, Art. 396.

‡ Let TP, TQ be tangents to an ellipse, and OAB, OCD chords parallel to TP, TQ. Determine a diameter of each of the two parabolas through A, B, C, D (Ex. 184); then it is easily seen that PQ and the diameter through T in the ellipse are parallel to the diameters of the parabolas.

§ See NEWTON'S *Principia*, Lib. I. sect. V. lemma 19. The next ten examples are mostly solved in the same Section, which will repay a careful study. See also Book II. of Leslie's *Geometry of Curve Lines*.

B and C; the straight lines joining any point on the curve to B and C will meet PS and PQ in points T and R such that PR varies as PT, and conversely. Hence shew how to draw a tangent to the conic at any point; and shew how to draw a conic through five given points, and prove that one conic only can be so drawn.

366. If two straight lines BM, CM turn about fixed poles B and C so that their intersection M moves along a fixed straight line or directrix, and if BD and CD be drawn at given angles to BM and CM respectively, the point of concourse D of the second pair of lines will trace a conic passing through the poles B, C, and conversely.[*]

367. By the foregoing construction (or otherwise) determine any number of points on a conic passing through five given points.[†]

368. Describe a conic passing through four points, three points, two points, or one point, and likewise touching one, two, three, or four straight lines respectively. Examine the cases in which two or more of the given points or lines coalesce.

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[*] For distinctness of conception let the points B, D, A, P in the following solution be supposed to lie (in the order specified) on one branch of a hyperbola and C on the other branch, as in NEWTON'S figure (lemma 21). Now when the moving point M has an assumed position N on the *directrix* let P be the corresponding fixed point on the locus of D. Draw BDT, CDR through any other position of D, and make the angle BPT equal to BNM and CPR equal to CNM; then it may be shewn that $PT : NM = PB : NB$, and $PR : NM = PC : NC$. Hence PT varies as PR, and therefore by the preceding lemma (Ex. 365)—as NEWTON abruptly concludes—the locus of D is a conic through the points B, C, P. The last step (see Le Sueur and Jacquier's edition of the *Principia*) is explained as follows: when NM becomes infinite, let D assume the position A; then it may be proved that BA is parallel to PT and CA to PR. Let PT, CA meet in S and PR, BA in Q. Then $ASPQ$ is the parallelogram of Ex. 365, and PT has been shewn to vary as PR.

[†] Still using the same figure, let A, B, C, P, D be the given points. Take ABC and ACB as the given angles which are to rotate about B and C as poles. The other two points P and D enable us to determine two points M and N on the *directrix*, and the whole curve can then be described. Or again, if A, B, C, D, E be five points on a conic, let AC, BE meet in F, and draw from D a parallel to CA to meet BE in G; then to determine the point H in which the parallel meets the curve again, we have $DG.GH : BG.GE = AF.FC : BF.FE$. The diameter bisecting the parallels can now be drawn, and in like manner a second diameter can be determined. Hence the centre is known, &c.

369. If AFP, BQG be the tangents to a conic at the ends of a diameter AB, and FG and PQ be tangents to the same and intersect in O, shew that

$$AF : BQ = FP : GQ = FO : OG,$$

and that PG, FQ, AB cointersect.

370. If in a parallelogram $LMIK$ any conic be inscribed touching the sides ML, IK, KL, IM in A, B, C, D, any fifth tangent to the conic will meet those sides in points F, Q, H, E such that

$$ME : MI = BK : KQ,$$

and $$KH : KL = AM : MF.$$

371. If the inscribed conic in the preceding example be fixed,

$$KQ \,.\, ME = KH \,.\, MF = \text{a constant.}$$

Moreover, if a sixth tangent be drawn to meet DE in e and IQ in q, then

$$KQ : Me = Kq : ME = Qq : Ee.$$

372. Hence shew that the diagonals of the quadrilateral $EqQe$ are bisected by one and the same diameter of the conic, and that the locus of the centre of a conic inscribed in a given quadrilateral is a straight line bisecting its three diagonals.

373. If IB, ID be the tangents at given points B, D of a conic, and EQ the intercept made by them on any other tangent to the same, shew that $IE.IQ$ varies as $ED.QB$.[*] Hence, if from two fixed points in a given pair of straight lines any other two lines be drawn, each to meet the opposite fixed line, shew that if the straight line joining the points of concourse envelopes a conic touching the fixed lines at the fixed points, the locus of the intersection of the variable lines will be a conic satisfying the same condition, and conversely. Examine the case in which the fixed lines are parallel.

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[*] By making the sixth tangent in Ex. 371 coincide successively with IE and IB we deduce that the two rectangles are as MI to MD. When M is at infinity they become equal (Ex. 183).

374. Given five tangents to a conic, determine its five points of contact with them;* and given five points on a conic, determine the five tangents thereat.

375. Given the centre of a conic and a self-conjugate triangle, shew how to determine the diameters conjugate to its sides and to describe the curve. Hence (or otherwise) shew how to describe a conic touching five given straight lines.

376. Given a tangent to an ellipse, its point of contact, and the director circle, shew how to construct the ellipse.

377. Two ellipses have a common focus and equal major axes, and one of them revolves about this focus in its own plane whilst the other remains fixed: prove that their chord of intersection envelopes an ellipse confocal with the fixed ellipse.

378. The condition that a straight line which makes intercepts CB and CD on two fixed straight lines should envelope a conic touching the fixed lines is of the form

$$a . CB . CD + b . CB + c . CD + d = 0,$$

where the ratios of a, b, c, d are constant. Determine the points of contact of the envelope with the fixed lines; and explain the result when the intercepts are connected by a relation of the form

$$CB . CD = \text{a constant.}$$

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* If $ABCD$ be the pentagon formed by the five tangents, the straight line joining D to (AC, BE) passes through the point of contact of AB, as appears most simply by supposing two sides of the enveloping hexagon in Brianchon's theorem to coalesce. When five points are given, the tangents thereat may be drawn and number of points on the curve may be found with the help of Pascal's hexagon. See Salmon's *Conic Sections*, Art. 269.

† Call the straight line bisecting the three diagonals of a quadrilateral its DIAMETER. The diameters of any two of the quadrilaterals formed by the five tangents determine the centre of the conic, and any one of the quadrilaterals gives a self-conjugate triangle. For another solution, in which the five points of contact are first found, see LIB. I. Sect. V. of the *Principia* (prop. 27, prob. 19); and see Ex. 374 (note), and Besant's *Conic Sections treated Geometrically*, Art. 229 (1875). It is evident that the diameters of the five quadrilaterals formed by five straight lines meet in one point, viz. the centre of the conic touching the five lines.

379. In order that the envelope in Ex. 378 may be a parabola the ratio of a to the other constants must vanish.* Hence shew that the polars of a fixed point with respect to a series of confocal conics, and likewise the normals appertaining to the tangents drawn to them from that point, envelope a parabola touching the axes of the confocals.

380. If OP and OQ be the tangents from a fixed point O to any conic which has two given points for foci, each of the corresponding normals is the polar of O with respect to a conic having the same foci; and the circle about OPQ passes through a second fixed point F, such that CF and CO lie on opposite sides of the transverse axis and make equal angles therewith, and $CF.CO = CS^2$.

381. A tangent being drawn from an extremity of one axis of an ellipse to a coaxal ellipse, find the length of its intercept on the other axis and the ordinate of its point of contact to either axis.

382. Deduce from Ex. 356 that, if a single n-gon can be described about a given conic and inscribed in a given confocal, any number of n-gons can be so described.

383. If a triangle can be circuminscribed to two confocal ellipses, the straight lines joining the extremities of the axes of the outer must pass through the intersections of the tangents at the extremities of the axes of the inner ellipse.†

384. If PQR be a triangle circuminscribed to a pair of confocal ellipses and P' be the point of contact of QR, shew that the confocal hyperbola through P passes through P' and the

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* The general condition of Ex. 378 is implied in Ex. 373, and the condition that the envelope may be a parabola is inferred from Ex. 183. In what follows, supposing h, k to be the coordinates of the fixed point, we see from Ex. 343 (or Ex. 270) that, if the enveloping line make intercepts CL and CM on the transverse and conjugate axes, $h.CL - k.CM = CS^2$; and consequently that the envelope is a parabola which makes intercepts $\frac{CS^2}{h}$ and $-\frac{CS^2}{k}$ on the axes.

† The proof may be simplified by considering the special case in which a side of the triangle is parallel to an axis of the ellipses. The semi-axes a, b and a', b' of the outer and inner ellipses are connected by the relation $\frac{a'}{a} + \frac{b'}{b} = 1$.

point diametrically opposite thereto, and that if the outer ellipse be regarded as traced by means of a loop $PP'P$ passed round the inner, the loop will be bisected at P'.

385. The area of an ellipse is a mean proportional to the areas of its auxiliary circles.

386. A quadrilateral can be circuminscribed to two confocal ellipses if the common difference of the areas of their major and their minor auxiliary circles be equal to the area of the inner ellipse; the locus of the pole of any chord of the outer ellipse which touches the inner is a circle whose diameter is equal to the sum of the axes of the latter; the tangents to the inner from any point on the outer ellipse are as the parallel focal chords of the latter; the chord joining the ends of a pair of semi-axes of the outer touches the inner ellipse and is divided at its point of contact into segments equal to the semi-axes of the latter.

387. Prove Fagnani's theorem, that a quadrant of an ellipse can be divided into segments which differ by the difference of its semi-axes, the greater segment being that which is terminated by the minor axis.*

388. If C be the common centre of an ellipse and a circle of equal area, P the point in which the circle meets a quadrant $AQPB$ of the ellipse, and CQ be equal to radius conjugate to CP; shew that the middle point of the quadrantal arc AB lies within the arc PQ.

389. If a hexagon can be circuminscribed to two confocal ellipses, and AP, BQ be the tangents to a quadrant $A'B'$ of the inner from the extremities of the semi-axes CA, CB of the outer ellipse, and F be Fagnani's point of division of the quadrantal arc AB, shew that

$$\text{arc } B'Q - \text{arc } A'P = \text{arc } BF - \text{arc } AF = CA - CB.\dagger$$

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* The point of contact last mentioned (Ex. 386) divides the inner ellipse in the manner specified. For another geometrical proof see Salmon's *Conic Sections*, Art. 400.

† If $\lambda^2 = CA^2 - CA'^2 = CB^2 - CB'^2$, and if the tangent at B' meet the outer ellipse in O, it may be shewn that $\frac{1}{\lambda} = \frac{1}{CA} + \frac{1}{CB}$; $B'O = CA - \lambda$; $OP = \lambda$; $AO = CB$, &c.

390. Any circle through the focus S and the further vertex A' of a hyperbola whose eccentricity is *two* meets the curve in three points P, Q, R which determine an equilateral triangle,[*] and conversely the circumscribing circle of any equilateral triangle inscribed in a hyperbola whose eccentricity is two passes through a focus and the further vertex; the focal vectors SP, SQ, SR meet the curve in three other points which likewise determine an equilateral triangle; if P be any point on the S-branch of the curve the angle $A'SP$ is double of the angle $SA'P$, and if Q be any point on the opposite branch the supplement of $A'SQ$ is double of the supplement of $SA'Q$; any chord through S subtends a right angle at A'; the equilateral triangle PQR envelopes a fixed parabola having S and the S-directrix for focus and directrix; the tangents to the hyperbola at P, Q, R form a triangle $P'Q'R'$ inscribed in a fixed hyperbola of eccentricity four; the tangents to the latter at P', Q', R' form a triangle inscribed in a fixed hyperbola of eccentricity eight, and so on continually.

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[*] This hyperbola—whose directrix bisects SA'—is one of the TRISECTORS (Ex. 308) of any circular arc whereof $A'S$ is the chord; and the meaning of the remarkable property that PQR is an equilateral triangle is that *the problem of bisecting a given angle* α *admits of the three distinct solutions* $\frac{1}{2}\alpha$, $\frac{1}{2}(2\pi \pm \alpha)$. Since the solution must in any case be threefold, it is evident *a priori* that it cannot be effected by means of a straight line and circle, which can intersect in two points only. All this is fully pointed out by BOSCOVICH in his *Sectionum Conicarum Elementa*, §§ 274–279. NEWTON shewed (*Arithmetica Universalis*, prob. 36) that the locus of the vertex of a triangle on a given base and having one base angle differing from twice the other by a constant angle is a cubic curve, which reduces to the hyperbola in question when the constant angle vanishes; and he remarked that (P being a point on the S-branch) the angle at A' in the triangle $A'SP$ is equal to ONE THIRD of the exterior angle at P.

NOTE.

The undermentioned Examples and others are solved wholly or in part in vols. i–xxix of *Mathematical Questions with their Solutions from the* EDUCATIONAL TIMES (London, 1864—78):

Ex. 79 (vol. xxii.); 174, 222, 226 (i.); 324 (xxii.); 328 (ii.); 331, 332 (iii.); 334 (xii.); 336 (xiii.); 338 (xxi.); 339, 340 (xxii.); 341 (xxvi.); 347, 348 (iv.); 361 (xiii.).

CHAPTER V.

THE ASYMPTOTES.

53. THE *Asymptotes* of a hyperbola are two diameters equally inclined to the axis and such that, if E be any point on either of them and CN its central abscissa, then

$$EN : CN = CB : CA;$$

in other words, the asymptotes are the diagonals of a certain rectangle which is determined by the two axes of the hyperbola.

If any two conjugate diameters meet EN and its prolongation in L and M, it follows from Art. 44 and the above relation that

$$NL.NM : CN^2 = CB^2 : CA^2 = EN^2 : CN^2;$$

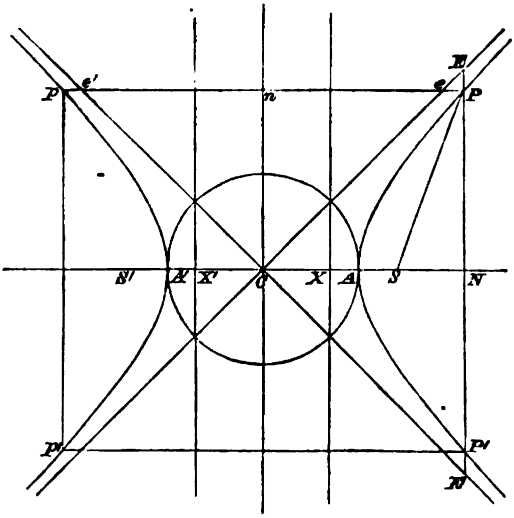
and hence that in the limiting case in which the diameter CL coalesces with an asymptote CE its conjugate CM coalesces with the same, or *an asymptote may be regarded as a diameter conjugate to itself.*

Two hyperbolas are said to be conjugate when the tranverse axis of each is coincident with the conjugate axis of the other; thus, the transverse and conjugate axis of a hyperbola being AA' and BB', those of the *Conjugate Hyperbola* will be BB' and AA'. It is evident that a pair of conjugate hyperbolas have the same asymptotes but lie on opposite sides thereof.

54. *Limiting positions of Tangents.*

The asymptotes are so called because, being produced, they continually approximate to the curve (Art. 56) but without actually meeting it until produced infinitely. We shall shew that such lines may be regarded as *tangents whose points of contact are at infinity.*

The tangent at any point P meets the axis in a point T such that CT varies inversely as CN the abscissa of P (Art. 39) and therefore vanishes when CN is infinite. To determine the position of a pair of tangents which pass through the centre of the hyperbola, draw the tangents from S to the Auxiliary Circle (Art. 6), and draw the diameters through the points in which they meet the S-directrix. The points in question will lie on the circle, since the diameter AA' of the circle is divided harmonically at S and X, and the directrix is therefore the polar of S with respect to the circle. The tangents to a hyperbola from its centre are therefore those diameters which pass through *the intersections of the directrices with the Auxiliary Circle.*



It is easily seen that the said diameters possess the property

$$EN : CN = CB : CA,$$

and are therefore identical with the asymptotes.

It is likewise evident that they possess the property,

$$CN : CE = CS : CA = \text{the eccentricity};$$

and hence that all hyperbolas which have the same (or parallel) asymptotes and lie on the same sides thereof are SIMILAR conics; and *the asymptotes* themselves taken together are the limiting form to which the curve tends when its axes are diminished indefinitely, and they *may be regarded as constituting a similar hyperbola.*[*]

The hyperbola may be called *Acute* or *Obtuse* according as the interior angle between its asymptotes is less or greater than a right angle; that is to say, according as its conjugate axis is less or greater than its transverse axis. In the intermediate case, when they are equal, it is called *Rectangular.*

It is easily seen that any two tangents to the same branch of the hyperbola intersect within the interior angle between the asymptotes and themselves contain a greater angle; and likewise that any two tangents to opposite branches contain an angle less than the exterior angle between the asymptotes; and hence that *an obtuse hyperbola can have no real tangents at right angles.*

55. *A construction for the Normal.*

If P be a point on the curve whose ordinate to either axis meets the nearer asymptote in E, and if the normal at P meet the axis in G, then (Art. 43, Cor.) in the case of the transverse axis,

$$CN.NG : CN^2 = CB^2 : CA^2 = EN^2 : CN^2;$$

and therefore $CN.NG$ is equal to EN^2 and the angle CEG is a right angle, as may be shewn likewise for the case of the conjugate axis. Hence the following construction for the normal at a given point P:

Let the ordinate of P to either axis meet the nearer asymptote in E, and through E draw a perpendicular to CE to meet the same axis in G; then will PG be the normal at P.

When CN is infinite the normal itself coincides with EG and is perpendicular to the asymptote.

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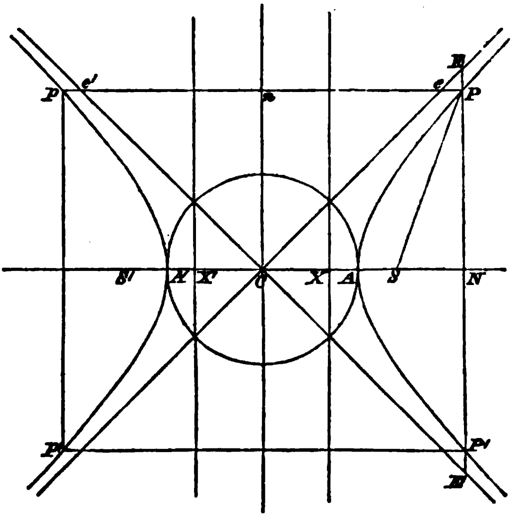
[*] Notice in Art. 38 that when SY *touches the circle* its diameter through Y should be the tangent to the hyperbola; and also that in this case $SY = CB = HZ$ Moreover (Art. 14, Cor. 1) the diameter conjugate to $C\infty$ must be parallel to the tangent at ∞, and must therefore coalesce with $C\infty$ itself. The HYPERBOLA is distinguished as the conic which has a pair of tangents whose points of contact are at infinity and whose chord of contact is therefore the *Straight Line at Infinity* (Art. 17, Cor. 2); and the PARABOLA is distinguished as the conic to which the line at infinity is a tangent, since (Art. 27) $SY^2 = SA.SP = SA.ST$, and therefore when SP becomes infinite the tangent TY is removed wholly to infinity.

PROPOSITION I.

56. *If a parallel to either axis of a hyperbola be drawn through any point on the curve to meet the asymptotes, the product of its segments between the point and the asymptotes will be equal to the square of the semi-axis to which it is parallel.*

First let a principal double ordinate PP' be produced to meet the asymptotes in E and E'; then will $PE.PE'$ or $P'E.P'E'$ be equal to CB^2.



For by Art. 33 and by a property of the asymptotes, if PP' meet the transverse axis in N,

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2 = EN^2 : CN^2,$$

or
$$PN^2 + CB^2 = EN^2;$$

and therefore

$$PE.PE' = P'E.P'E' = EN^2 \sim PN^2 = CB^2.$$

In like manner it may be shewn that CA is a mean proportional to the segments Pe and Pe' of a straight line drawn through P parallel to the transverse axis to meet the asymptotes.

Hence it appears that the distance of P (or P') from the nearer asymptote varies inversely as its distance from the other, and when the latter distance is increased indefinitely the former is diminished indefinitely; the curve therefore as it branches out continually approximates to its asymptotes, but without actually meeting them at any finite distance from the centre.

It is easy to shew by a *reductio ad absurdum* that no diameter other than CE or CE' have the above property, or in other words, that the hyperbola cannot have more than two asymptotes.

Corollary.

If PO be drawn parallel to one asymptote $E'C$ to meet the other asymptote EC, then will PO vary as PE and CO as PE', and therefore $PO.CO$ (fig. Art. 60) will be constant; and it may be shewn by taking P at the vertex that it is equal to $\frac{1}{4}CS^2$ or $\frac{1}{4}(CA^2 + CB^2)$.

PROPOSITION II.

57. *The intercepts on any tangent to a hyperbola between the curve and its asymptotes are equal to one another and to the parallel semi-diameter; and the opposite intercepts on any chord between the curve and its asymptotes are equal to one another.*[*]

(i) Let the tangent at P, supposed parallel to the semi-diameter CD, meet any two conjugate diameters in L and L'; then by Art. 47

$$PL.PL' = CD^2.$$

Hence in the case in which L' coalesces with L and CL therefore becomes an asymptote, PL^2 is equal to CD^2; and in like manner, if the same tangent meet the other asymptote in M, PM^2 is equal to CD^2.

Therefore $$PL = PM = CD.$$

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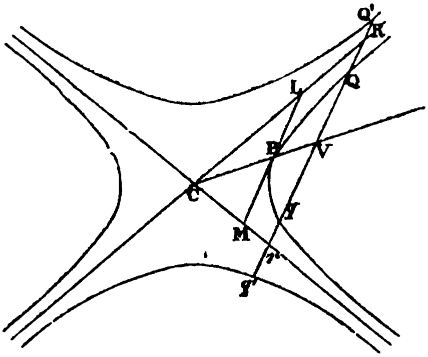
[*] The hyperbola and its *Asymptotes* being similar conics (Art. 54), the above is a special case of Ex. 50. The latter follows at once from Art. 14 since, when the *direction* of CX and the *magnitude* of $CS : CX$ are given, if the direction of CV be assumed that of SV (which is perpendicular to the ordinates of CV) is determined. It is evident that a pair of conjugate hyperbolas also make equal intercepts QQ', qq' on any chord.



(ii) Next let Qq be any chord of the hyperbola meeting the asymptotes in R, r, and let LPM be one of the tangents to which it is parallel.

Then the diameter CP bisects the chord Qq, and from above it is evident that it likewise bisects Rr; whence it follows that

$$QR = qr, \text{ and } Qr = qR.$$

PROPOSITION III.

58. *The product of the segments into which any chord of the asymptotes is divided by either of the points in which it meets the curve is equal to the square of the parallel radius.*

(i) Using the same construction as in Prop. II., let V be the middle point of the chord Qq. Then by Art. 34 and by parallels (first taking the case in which Q and q lie on the same branch of the curve),

$$\begin{aligned} QV^2 + CD^2 : CV^2 = CD^2 : CP^2 &= PL^2 : CP^2 \\ &= RV^2 : CV^2, \end{aligned}$$

or

$$QV^2 + CD^2 = RV^2.^*$$

Hence $RQ \,.\, Qr = Rq \,.\, qr = RV^2 - QV^2 = CD^2$,

or CD is a mean proportional to RQ and Qr, and to Rq and qr.

The above proof may be adapted to the case in which Q, q

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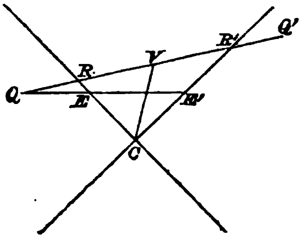
* Conversely, if this relation be assumed the point R must always lie on one of two straight lines which continually approach the curve, that is to say, on one of the asymptotes.

lie on opposite branches of the hyperbola by writing $-CP^2$ for CD^2 and $-CD^2$ for CP^2.

(ii) These results may also be deduced as follows from Prop. I.

From any point Q on the curve draw QRR' in any given direction to meet the asymptotes, and draw QEE' parallel to



the transverse or the conjugate axis to meet the same. Then QR varies as QE and QR' as QE', and therefore, $QE.QE'$ being constant, $QR.QR'$ is likewise constant.

Supposing RR' to become a diameter or a tangent, according as its direction cuts both branches of the curve or one only, we see that $QR.QR'$ is equal to the square of the parallel semi-diameter or of the intercept on the tangent between the curve and either asymptote.

Corollary 1.

If the tangent at P meet the asymptotes in L, M (fig. Art. 60), and if O be the middle point of CL, and OP be therefore parallel to CM, then (Prop. I. Cor.)

$$CL.CM = 2CO.2PO = CS^2;$$

and therefore the area LCM is constant, that is to say, *the area of the triangle bounded by the asymptotes and any tangent is of constant area*, and it is equal to $CA.CB$. It is otherwise evident that the triangle in question is one-fourth of the conjugate circumscribing parallelogram (Art. 46).

Corollary 2.

Moreover, if PK be drawn perpendicular to CL,

$$CP^2 \sim PL^2 = CK^2 \sim LK^2 = 2CO.2OK;$$

where OK varies directly as OP and therefore inversely as CO. Hence another proof that *the difference of the squares on two conjugate diameters is constant* (Art. 45).

Corollary 3.

So long as the chord RQr is drawn in a specified direction Qr varies inversely as QR. If the chord be taken constantly *parallel to the asymptote* CM, so that the point r recedes to ∞, it follows that $Q\infty$ varies inversely as QR, or directly as CR; and moreover, that if QO be any finite portion of the chord, then $O\infty$ likewise varies as CR, and the rectangle $QO.O\infty$ varies as $CR.QO$. If YOZ be a chord drawn in any other specified direction and meeting the chord parallel to the asymptote in O, then (Art. 16, Cor. 1) $OY.OZ$ varies as $QO.O\infty$, or as $CR.QO$; and in the special case in which $Q\infty$ is a fixed chord $OY.OZ$ varies as the length QO.

SCHOLIUM A.

If the hyperbola be defined as the locus of a point P such that if O be its projection upon one of two fixed straight lines CL, CM (the asymptotes) by a straight line parallel to the other, $CO.PO =$ a constant c^2, we may proceed to investigate the properties of the curve as follows.

If LM be drawn in a *specified* direction through any position of the tracing point P, it is evident that $PL.PM$ is constant, and also that in the case in which LM becomes a tangent it is bisected at its point of contact P. In this case $CL.CM = 2CO.2PO = 4c^2$, and the triangle CLM is of constant area. It may now be shewn that (with the notation of Art. 57) $QR = qr$; CP bisects Qq and all other chords parallel to the tangent at P; QV^2 varies as CV^2-CP^2; and that the difference of the squares of any two conjugate diameters is constant (Art. 58, Cor. 2).

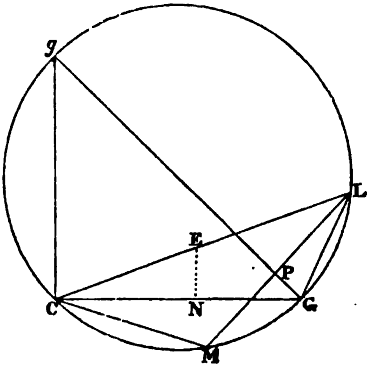
A straight line parallel to either asymptote CM meets the curve in one point only, since (figure of Art. 60) if CO be supposed constant, $CO.PO$ vanishes when P is at O, and increases continuously up to ∞ as P recedes from O, and is therefore equal to c^2 for one position of P only. Hence at any point P between the curve and its asymptotes $CO.PO$ is less than c^2. Moreover, for any assumed position of the intercept LM it is evident that $PL.PM$ is a maximum, and therefore $PO.CO$ is a maximum, when $PL = PM$. Hence at the point of concourse P' of any two tangents to the same branch of the curve $P'O\ CO$ is less than c^2, and P' therefore lies between the curve and its asymptotes, or the curve is *convex* to its asymptotes.

Lastly, if the hyperbola be regarded as the envelope of a straight line LM which contains a triangle of constant area with two fixed straight lines CL and CM, it may be shewn by the following method (which applies also to Ex. 362) that LM is bisected at its point of contact. If P be the point of concourse of the tangent line in any two positions LM, $L'M'$, the areas LPL' and MPM' are equal, and $PL.PL' = PM.PM'$; and therefore in the case in which $L'M'$ and LM coalesce, $PL^2 = PM^2$, or LM is bisected at the point P of the envelope. The hyperbola may also be regarded as a special case of the envelope in Ex. 378, which makes intercepts $\frac{-d}{b}$ and $\frac{-d}{c}$ upon the fixed tangents, and therefore touches them at *infinity* when b and c vanish.

PROPOSITION IV.

59. *Any tangent and its normal meet the asymptotes and the axes respectively in four points lying on a circle which passes through the centre of the hyperbola.*

The circle whose diameter is the intercept Gg made by the axes on any normal passes through the centre, since the angle gCG is a right angle.



Let this circle meet the asymptotes in L, M, and let LM meet Gg in P. From any point E in CL draw EN perpendicular to CG.

Then $\angle ECN = GCM = GLP$, in the same segment,

and $\angle CEN = ECg = LGP$, in the same segment;

and therefore the triangles CNE, LPG are similar, so that the angle at P is *a right angle*, and

$$PG : PL = EN : CN = CB : CA.$$

Similarly $$PL : Pg = CB : CA.$$

Hence $$PG : Pg = CB^2 : CA^2,$$

or P is the point at which Gg is normal to the curve (Art. 52, Cor. 2); and LM, which is at right angles to Gg, is the tangent at P.

Corollary.

From this construction it appears again the tangent LM is bisected at its point of contact; and that

$$PG : CD = CD : Pg = CB : CA,$$

where CD is the semi-diameter parallel to the tangent.

PROPOSITION V.

60. *The diameters of a hyperbola being regarded as terminated at the points in which they meet the curve or its conjugate, any two conjugate diameters are the diagonals of a parallelogram whose sides are parallel to and are bisected by the asymptotes, the tangents at their extremities meet on the asymptotes, and the difference of their squares is constant.*

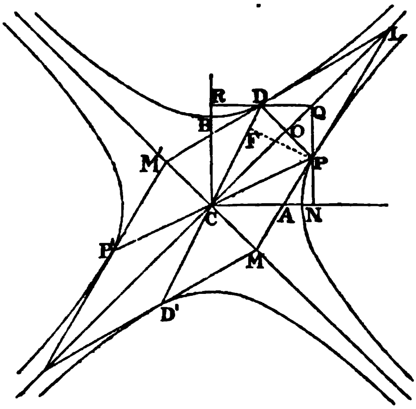
(i) From a point L on either asymptote of a pair of conjugate hyperbolas let a tangent be drawn to each, the one tangent meeting its curve in P and the second asymptote in M, and the other meeting its curve in D and the second asymptote in M'. Then will CP, CD be conjugate semi-diameters, and PD will be parallel to MM', and will be bisected at the point O in which it meets CL.

For since (Art. 57) the tangent LM is bisected at P, and LM' at D, therefore *PD is parallel to the asymptote MM'*, and it also bisects CL.

Moreover (Art. 56, Cor.),

$$PO.CO = \tfrac{1}{2}(CA^2 + CB^2) = DO.CO,$$

or PO is equal to DO; that is to say, *PD is bisected by the asymptote CL.* But PD likewise bisects CL, and therefore CD



is parallel to the tangent at P and is conjugate to CP. And if the parallel tangents touch the curves in P' and D', as in the diagram, the one will evidently pass through M' and the other through M; and $P'D$ will likewise be parallel to one asymptote and bisected by the other.

(ii) Lastly, if PN be the ordinate of P to the transverse axis, and if it meet CL in Q, it is easily seen that $OQ = OP$. And in like manner the ordinate DR to the conjugate axis meets CL in a point Q' such that $OQ' = OD = OP = OQ$; that is to say, it meets it in the same point Q.

Hence $\quad CQ^2 - CP^2 = QN^2 - PN^2 = CB^2,$

and $\quad CQ^2 - CD^2 = QR^2 - DR^2 = CA^2;$

and therefore $\quad CP^2 - CD^2 = CA^2 - CB^2.$

Corollary.

To describe a pair of conjugate hyperbolas with given straight lines CP and CD as conjugate semi-diameters: draw CO to the middle point of DP and draw CM parallel to DP; then will CM and CO be the asymptotes, and the axes will be the bisectors of the angles between them, and the foci will be the points in which the axes are cut by a circle whose radius CS is a mean proportional to PO and $2\,CO$.

SCHOLIUM B.

Conjugate Hyperbolas are by no means to be regarded as organically related and together making up one continuous curve; but the one is a sort of auxiliary curve to the other, as the circle on its major axis is, for example, to the Ellipse.

(i) The two branches of a single hyperbola are to be regarded as constituting one continuous curve as was pointed out on p. 10, and as may be further illustrated in the following way. Let the hyperbola be considered to be traced by the extremity P of a focal vector SP (see fig. p. 145) moving round in the direction of the hands of a watch from the initial position SA. As SP turns through an infinitesimal angle its extremity passes to a *consecutive* point on the curve, till at length by the continuous rotation of the focal vector the point P recedes to infinity, SP having become parallel to the asymptote CE: it then passes instantaneously to the *opposite* position at infinity, that is to say (SP revolving gradually as before) the point P passes at once from the extremity of the line $CE\infty$ to the extremity of the line $EC\infty$: at this infinitely distant point the curve *crosses its asymptote*, and P proceeds to trace the opposite branch in the direction $p'A'$, and so forth. The two infinitely distant extremities of an asymptote or of any straight line may therefore be regarded as consecutive points, which likewise results from considering any straight line as (1) a circle of infinite radius in its own plane, or (2) as one of the great circles of a sphere whose radius has become infinite. Carrying on the latter illustration, we see that (since the length of a great circle on any sphere is constant) any finite straight line in a given plane together with its complement (p. 77) may be regarded as making up a *constant* infinite length; as was implicitly assumed in Chap. IV. Scholium C (p. 102), for if the bifocal property of the hyperbola,

$$HP - SP = AA',$$

be equivalent to $\quad H\infty P + SP = A\infty A',$

then $\quad HP + H\infty P = AA' + A\infty A'.$

(ii) It may be useful at this stage to give a conspectus of the several ways of viewing those diameters of the hyperbola which are not geometrically terminated by the curve.

a. By introducing the conception of *imaginary points* we may treat the hyperbola as a quasi-ellipse, and ignore the distinction between intersecting and non-intersecting diameters of the curve.

b. If we assign certain real magnitudes to the non-intersecting diameters (Art. 34)—arbitrarily, indeed, but in accordance with a partial analogy—we may then proceed to shew (Art. 57) that any such diameter is equal to the intercept on the parallel tangent made by the asymptotes, and may prove as in Art. 58, Cor. 2 that the difference of the squares of any two conjugate diameters is of constant magnitude.

c. The non-intersecting diameters may also be treated as terminated by the conjugate hyperbola, as in Arts. 45 and 60. The objection to this mode of treatment is that it not only proceeds upon an artificial analogy but tends to obscure the fact of the essential continuity and oneness of the two branches of the hyperbola.

d. Another method—very simple in practice, but presenting of course the same difficulties at the outset—is to start with the Equilateral Hyperbola (some of the properties of which can be proved in terms applicable likewise to the circle or Equilateral Ellipse) and to transfer the results thus obtained to the general hyperbola by the method of Orthogonal Projection.

(iii) APOLLONIUS, in Lib. I. prop. 14 of his Conics, defines the two branches of a hyperbola as *Opposite* Sections (Ἀντικειμέναι). At the end of the same book (prop. 56) he shews, quite independently of the asymptotes, how to construct two pairs of opposite sections with one and the same given pair of conjugate diameters, and he defines the curves so drawn as *Conjugate* (Συζυγεῖς). He afterwards proves in Lib. II. prop. 15 that opposite sections have the same asymptotes, and in Lib. II. prop. 17 that conjugate opposite sections have the same asymptotes. The term *Conjugate* has also been sometimes applied to the two branches of a single hyperbola, as for example in prob. 36 of the *Arithmetica Universalis*, where the words "conveniant ad *conjugatam* Hyperbolam" refer to the further branch.

EXAMPLES.

391. The eccentric circle of any point with respect to a hyperbola cuts the directrix at two points lying upon radii which are parallel to the asymptotes. Trace the hyperbola by the method of Art. 4, shewing that the two points in which the circle cuts the directrix correspond to the points at infinity upon the asymptotes, and the segment of the circle beyond the directrix to the further branch of the hyperbola.

392. The circle described about either focus of a hyperbola so as to bisect the semi-latus rectum cuts the hyperbola at points whose focal distances are parallel to the asymptotes; and the concentric circle which touches the asymptotes has its diameter equal to the conjugate axis.

393. Express the eccentricity of a hyperbola as a function of the angle between its asymptotes. If the eccentricity and two points on the curve be given, and if one asymptote pass through a third fixed point in the same straight line with the former two, the locus of the centre will be a circle.

394. If the abscisses upon either asymptote of any number of points on a hyperbola be in arithmetical progression, their ordinates will be in harmonical progression, and conversely.

395. The ordinates to either asymptote of the extremities of any chord of a hyperbola and the point of contact of a parallel tangent are proportionals.

396. The intercept made by the directrices of a hyperbola on either asymptote is equal to the transverse axis.

397. A hyperbola being regarded as the locus of a point whose distance from a given point is equal to its distance from a fixed straight line estimated in a given direction, prove that the given direction is that of an asymptote. Shew also that the straight line drawn from a focus to the nearer directrix parallel to an asymptote of a hyperbola is equal to the semi-latus rectum and is bisected by the curve.

398. The distance of any point on a hyperbola from either focus is equal to the intercept on either asymptote between the ordinate of the point and the corresponding directrix. Hence prove in Art. 60 that if S and H be the foci,

$$SP.HP = CQ^2 - CA^2 = CD^2.$$

Also prove that the difference of the distances of the ends of two conjugate radii of a pair of conjugate hyperbolas from their nearer foci is equal to the difference of the semi-axes.

399. Every chord drawn to a hyperbola from a fixed point on one asymptote is divided harmonically by that point and a fixed parallel to the said asymptote, and is bisected by a fixed parallel to the other.

400. The tangents at the vertices of a hyperbola meet its asymptotes on the circumference of the circle of which the straight line joining the foci is a diameter.

401. For what position of the tangent to a hyperbola is its intercept between the asymptotes a minimum?

402. The tangent to a hyperbola from the intersection of an asymptote with a directrix touches the curve upon a focal vector which is parallel to that asymptote.

403. The intercept on any tangent between the asymptotes subtends at the further focus an angle equal to half the angle between the asymptotes: it also subtends a constant angle at the intersection of the corresponding normal with either axis of the curve.

404. Every chord of a branch of a hyperbola which subtends at its focus an angle equal to the angle between the asymptotes touches a certain fixed parabola.

405. Find the relation between the intercepts made by any tangent to a hyperbola on two fixed straight lines parallel to the asymptotes.[*] If OA and OB be two straight lines given in position and AB the intercept which they make on any tangent to a fixed conic which touches them, deduce from Ex. 378 that the locus of the point P which completes the parallelogram $OAPB$ is a hyperbola whose asymptotes are parallel to OA and OB; and examine the case in which the fixed conic is a hyperbola having its centre at O. Also find the locus of Q if A and B be the points of contact of the given lines with any parabola which likewise touches a third given line.

406. The chords of intersection of any circle with the asymptotes of a hyperbola are equally inclined to either axis; the products of the segments of any two intersecting chords of the asymptotes are as the parallel focal chords; and if O be any

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[*] If ξ and υ be the reciprocals of the intercepts made by a variable straight line on two fixed axes, the general condition that the variable line should envelope a conic is that ξ and υ should be connected by an equation of the second degree. This system of "tangential coordinates" is fully developed in *A Treatise on some New Geometrical Methods* (vol. I., 1873) by the late Dr. James Booth, who had also given an account of his method in a separate tract published thirty years earlier. His discovery had however been anticipated by Prof. Plücker of Bonn, whose method given in Crelle's *Journal*, vol. VI. pp. 107—146 (1830), and dated Oct. 1829, is in reality identical with the above. See the obituary notice of Dr. Booth in the *Monthly Notices of the Royal Astronomical Society*, vol. XXXIX. pp. 219—225 (Feb. 1879).

point on a chord PQ parallel to the radius CD of the hyperbola, and L, M the points in which it meets the asymptotes, then

$$OL.OM \sim OP.OQ = CD^2.$$

407. The tangent to a hyperbola at P meets one asymptote in T and TQ is drawn parallel to the other to meet the curve in Q; prove that if PQ meet the asymptotes in L and M, the line LM will be trisected at P and Q.

408. The straight lines joining the points in which any two tangents to a hyperbola meet the asymptotes are parallel; and the intercepts which the tangents make upon the asymptotes are bisected by their chord of contact.

409. If one diagonal of a parallelogram whose sides are parallel to the asymptotes of a hyperbola be a chord of the curve, the other diagonal will lie upon the conjugate diameter; and conversely if the three sides of a triangle be taken as diagonals of three parallelograms whose sides are parallel to two given straight lines, their other three diagonals will pass through the centre of a hyperbola which circumscribes the triangle and whose asymptotes are parallel to the given lines.

410. In Art. 39, if CM and CN be the central abscissæ of the points in which the tangent meet the asymptotes, then

$$CV.CT = CM.CN = CP^2.$$

411. If the ordinate of a point on the hyperbola to a given diameter be equal to the conjugate semi-diameter, the product of the corresponding abscissæ will be equal to the square of half the given diameter.

412. Given the asymptotes of a variable hyperbola and a line parallel to one of them, if from the point in which it meets the curve a parallel to the other asymptote be drawn equal to either of the semi-axes, the locus of its extremity will be a parabola.

413. If an ellipse and a branch of a confocal hyperbola intersect in P and Q, the asymptotes of the hyperbola pass through the points on the auxiliary circle of the ellipse which correspond to P and Q.

414. A variable ellipse having its centre on a hyperbola and touching its asymptotes has in every position the maximum area: shew that its chord of contact with the asymptotes will envelope a similar hyperbola having the same asymptotes.

415. A parabola being drawn to touch the axes of a hyperbola at an extremity of each, prove that one asymptote is a diameter of the parabola and that the other is parallel to its ordinates.

416. If a parallelogram be formed by drawing two pairs of parallels to the asymptotes of a hyperbola, its sides will meet the curve at the extremities of two chords which intersect upon a diagonal of the parallelogram; and further, if any three hyperbolas have their asymptotes parallel, three and three, their three common chords will cointersect.

417. The tangents to an ellipse at P and Q being the asymptotes of a hyperbola, prove that a pair of their common chords are parallel to PQ, and that if the tangent to the hyperbola at an extremity of one of these chords pass through P the tangent at its other extremity will pass through Q.

418. With two conjugate diameters of an ellipse as asymptotes a pair of conjugate hyperbolas are drawn; prove that if one of them touch the ellipse the other will touch it, and that the diameters through the points of contact will be conjugate.

419. If from any point P on a hyperbola whose centre is C straight lines PM and PN be drawn parallel to and terminated by the asymptotes, and if an ellipse be drawn having CM and CN for conjugate radii, the direction conjugate to CP will be the same in both curves.

420. Given the base of a triangle and the difference of its base angles, or given the base of a triangle one of whose base angles is double of the other, it may be shewn that the locus of the vertex is a hyperbola. Determine the asymptotes and the eccentricity of each by supposing the vertex of the triangle to be at infinity.

421. If tangents be drawn to a hyperbola from any point on the conjugate hyperbola, their chord of contact will touch the opposite branch of the latter and be bisected at its point of contact.

422. The four normals to a hyperbola and its conjugate at points lying upon a perpendicular to either axis meet one another upon that axis.

423. Find the locus of the centre of gravity and the locus of the centre of the circumscribing circle of a triangle of constant area contained by one variable and two fixed straight lines.

424. A parabola and a hyperbola have a common focus and the asymptotes of the latter touch the former; prove that the tangent at the vertex of the parabola is a directrix of the hyperbola, and the tangents to the parabola where it meets the hyperbola pass through the further vertex of the latter.

425. Any two semi-diameters of a hyperbola contain the same area with the tangent at the extremity of either.

426. The asymptotes and any two conjugate diameters of a hyperbola divide any straight line harmonically.

427. The chords joining any point on a hyperbola to two given points on the same intercept a constant length on either asymptote; and the intercepts on a given parallel to an asymptote between the curve and two such chords are in a constant ratio.

428. If parallels to the asymptotes of a hyperbola be drawn from any point on the curve, any diameter will meet the parallels and either branch of the curve in three points whose central distances are in continued proportion.

429. If any two tangents to a hyperbola and their chord of contact intersect any parallel to either asymptote, the square of the intercept on the parallel between the curve and the chord of contact will be equal to the product of its intercepts between the curve and the tangents.

430. On a straight line drawn in a given direction to meet the three sides of a triangle a point is taken whose distances from the three sides are in continued proportion; prove that the locus of the point is a parabola or a hyperbola touching the two sides from which the extremes are measured at the extremities of the third side.

431. On a straight line drawn through a fixed point C to intersect two given straight lines a length CD is estimated a mean proportional to the intercepts between the fixed point and the two points of section; prove that the locus of D is a hyperbola whose asymptotes are the parallels through C to the fixed lines.

432. A diameter of a parabola and the tangent at its extremity being taken as the asymptotes of a hyperbola, what are the magnitudes to which the ordinate and abscissa of their point of concourse with respect to that diameter are a pair of mean proportionals? Conversely shew how to find a pair of mean proportionals to two given magnitudes.

433. The intercept on any parallel to an asymptote of a hyperbola (or to the axis of a parabola) between any point upon it and the polar of that point is bisected by the curve.

434. The intercept made upon any straight line through either vertex of a hyperbola by parallels drawn to the asymptotes through the other vertex is bisected at the point in which the straight line meets the curve again; the locus of the middle point of the intercept made upon any straight line through a fixed point by two given straight lines is a hyperbola to whose asymptotes they are parallel; and further, if the latter intercept be cut in any other constant ratio,* the locus of the point of section will still be a hyperbola. In what case will the eccentricity of the locus be independent of the ratio in which this intercept is divided?

435. If Q be a point on a hyperbola and N a point on the nearer asymptote, and if QE be drawn parallel to that

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* See the *Arithmetica Universalis*, prob. 25.

asymptote to meet the diameter conjugate to QN in E, then will the area of the quadrilateral $CEQN$ be equal to half the triangle cut off by any tangent from the asymptotes; and if the diameter parallel to QN meet QE in F and QI be drawn in the conjugate direction to meet the same asymptote in I, the quadrilateral $CIQF$ will have the same constant magnitude.

436. If O be any point in a chord QQ' of a hyperbola parallel to the tangent at P and CE an asymptote meeting that tangent in E, and if QR and OT be drawn parallel to the asymptote to meet the diameter which bisects the chord, prove that

$$QO.OQ' : PE^2 = \text{quadrilateral } QRTO : \text{triangle } CEP.$$

437. If P be any point on a hyperbola and CD be conjugate to CP, shew that a pair of straight lines PE, PF drawn parallel to the axes or to any other pair of conjugate diameters meet CD in points E and F such that

$$CE.CF = CD^2.$$

438. A parabola which has an asymptote of a hyperbola for one of its diameters meets the hyperbola in general in three points such that the ordinates of two of them to that diameter are together equal to the ordinate of the third.

439. From any point P on a hyperbola a parallel is drawn to one asymptote to meet the other in M, and an ellipse is drawn through P and M having its diameter which bisects PM parallel to the latter asymptote and in a constant ratio to its conjugate diameter, viz. in the ratio of PE to PM, where PE is a perpendicular to the latter asymptote; prove that the ellipse meets the hyperbola again in three points such that the distances of two of them from the latter asymptote are together equal to the distance of the third point from the same.*

440. If two ellipses touch a hyperbola and have its asymptotes for conjugate diameters, any straight line whose pole with

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* For Examples 425, 435—9 and others see De la Hire's *Sectiones Conicæ*, libb. IV., V. The references are given in detail in Walton's *Problems in illustration of the principles of Plane Coordinate Geometry*, pp. 276—292 (CAMBRIDGE, 1851).

respect to one of them lies on the hyperbola has it pole with respect to the other on the hyperbola.

441. If $ABCD$ be a convex quadrilateral, and AD be produced to K and BC to L so that KL may be parallel to AB, then will DL and CK be parallel to the asymptotes of a certain hyperbola described about the quadrilateral; and if $\alpha B\beta$, $\beta C\gamma$, $\gamma D\delta$, $\delta A\alpha$ be the sides of a parallelogram and be parallel to the asymptotes, the straight lines drawn from α, β, γ, δ to bisect AB, BC, CD, DA respectively will cointersect at the centre of the hyperbola.

442. If an ellipse pass through the centre and have its foci on the asymptotes of a hyperbola, and if the hyperbola passes through the centre of the ellipse, the axes of each curve are a tangent and normal to the other, and the two axes which are normals are of equal length.

443. If a diameter be taken at right angles to one asymptote of a hyperbola and parallels be drawn to the other asymptote from its extremities, any two supplemental chords from those points will make intercepts whose difference is constant upon the parallels.

444. The axes of the two parabolas which have a common focus and pass through two given points are parallel to the asymptotes of the hyperbola which passes through the common focus and has the given points for foci.

445. Any circle which touches both branches of a hyperbola makes an intercept equal to the transverse axis on either asymptote; the tangents to it where it meets the asymptotes pass through one or other of the foci, and those which pass through the same focus contain a constant angle equal to the angle between the asymptotes; and two of the chords of intersection of the circle with the asymptotes are tangents one to each of two fixed parabolas whose foci are at the foci of the hyperbola.

446. If two conjugate diameters of a hyperbola be equal, every two conjugate diameters must be equal and the asymptotes must be at right angles.

447. If two parallel chords of a conic meet any tangent to the same in T and t, and if any straight line meet the chords in O and o and the tangent in L, then

$$OT : ot = OL : oL.$$

Hence shew that the ratio of any two infinite parallel chords of a conic is finite, being a ratio of equality in the case of the parabola, and being equal to the ratio of the distances of the chords from the parallel asymptote in the case of the hyperbola.*

448. From two points O and o parallels are drawn to the asymptotes of a hyperbola, the parallels to one asymptote meeting the curve in M and m and the parallels to the other meeting the curve in N and n; shew that if

$$OM : ON = om : on,$$

the points O, o must lie either on one diameter or on a pair of conjugate diameters.

449. Prove by the Cartesian method or otherwise that if CA, CB and $C\alpha$, $C\beta$ be semi-axes of a fixed and a variable confocal ellipse respectively, P a point of contact of the latter with an ellipse drawn through the four extremities of the axes of the former, and PN the principal ordinate of P, then

$$CN : PN = CA . C\alpha : CB . C\beta.$$

Deduce that the locus of P is a hyperbola; and likewise determine its foci and asymptotes by considering special cases of the theorem.

450. Every ellipse drawn through the four extremities of the axes of a given ellipse is cut orthogonally by a hyperbola confocal with the given ellipse and having its equal conjugate diameters for asymptotes.†

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* See the notes on *Geometrical Evaluations* by R. W. Genese, M.A., in the *Messenger of Mathematics*, vol. IV. pp. 154—6 (1875).

† See Wolstenholme's *Mathematical Problems*, No. 1182 (ed. 2, 1878).

451. Find the locus of a point whose polar with respect to a conic cuts off a constant area from the space between two given conjugate diameters; and find the envelope of the polar of a point whose ordinates cut off a constant parallelogram from the same.

452. Having given the asymptotes of a hyperbola and a point on the curve, determine its foci and directrices.

453. Having given a focus and two points of a hyperbola and the direction of one of its asymptotes, or having given a focus and one point and the directions of both asymptotes, shew how to construct the curve.*

454. Given the centre of a hyperbola and three points on the curve, determine the directions of its asymptotes.

455. Having given the centre of a hyperbola and a self-conjugate triad, determine the directions of its asymptotes.

456. Having given four points and the eccentricity of a hyperbola, or four points and the direction of an asymptote, or three points and the direction of an asymptote and the eccentricity, shew how to construct the curve.

457. If three straight lines be drawn from three given poles and two of their intersections lie on fixed directrices, their third intersection will trace a curve of the second order. By the above system of radiants or otherwise describe a hyperbola having given one asymptote and three points or the directions of both asymptotes and three points on the curve.†

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* Five data in general determine a conic. An asymptote is equivalent to two data, viz. to a tangent and its point of contact or two coincident points on the curve: having given the direction only of an asymptote we have one of the two points at infinity on the curve: a focus will be seen to be equivalent to two conditions. Compare the note at the end of Salmon's *Conic Sections*, "On the Problem to describe a Conic under Five conditions."

† See Leslie's *Geometry of Curve Lines*, Book II. props. 10, 21, 22.

458. Having given four points on any conic and one point on its director, or having given four tangents to an equilateral hyperbola, shew how to construct the curve.[*]

459. The area of the sector of a hyperbola made by joining any two points on it to the centre is equal to the segment cut off from the space between the curve and its asymptotes by the ordinates of the same two points to either asymptote; any other two ordinates in the same ratio as the former[†] cut off an equal segment; and the segment cut off by any two ordinates is bisected by the ordinate which is a mean proportional to them. Prove also that if two equal hyperbolas have two of their asymptotes coincident and the other two parallel, any parallel to their common asymptote will cut off from the space between two adjacent branches produced to infinity an area equal to the parallelogram contained by the said parallel and the three asymptotes.

460. If O, P, Q, R... be any number of points on a branch of a hyperbola whose abscissæ CK, CL, CM, CN... on either asymptote are in continued proportion, the hyperbolic sectors OCP, OCQ, OCR... will be in arithmetical progression, and

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[*] The first case of Ex. 458 may be made to depend upon the second by reciprocating the conic with respect to the point on its director, as is done in GASKIN'S *The Geometrical Construction of a Conic Section subject to Five Conditions of passing through given points and touching given straight lines, deduced from the properties of Involution and Anharmonic Ratio, with a variety of general Properties of Curves of the Second Order*, p. 53 (Cambridge, 1852). It is in this very able tract that the term DIRECTOR seems to have been first used to denote the locus of intersection of tangents at right angles to a conic. The term is defined on p. 26, and in the Preface we read: "By a well known property of conic sections, the locus of the point of intersection of two tangents at right angles to one another is in general a circle concentric with the conic section, and when the curve is a parabola the locus is the directrix. There are several remarkable properties of this locus which, as far as the author is aware, have not been hitherto noticed, and he has found it convenient to denominate it the DIRECTOR of the conic section, which in the case of the parabola coincides with the directrix."

[†] It is easily seen that the four ordinates to either asymptote of the extremities of any two parallel chords are proportionals, and that the ordinate of the point of contact of any tangent is a mean proportional to the ordinates of the extremities of any parallel chord.

the segments *OKLP*, *OKMQ*, *OKNR*... will be in arithmetical progression. Given any three terms of a geometrical series and the logarithms of two of them, shew how to determine geometrically the LOGARITHM of the third.[*]

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[*] For the first part of Ex. 460 see GREGORII A. S. VINCENTIO *Opus Geometricum Quadraturæ Circuli et Sectionum Coni*, lib. VI. prop. 125, p. 594 (Antverpiæ, 1647), and his *Opus Geometricum Posthumum ad Mesolabium*, prop. 24, p. 252 (Gandavi, 1668). The second part may be solved by taking hyperbolic segments in arithmetical progression to represent the logarithms of a corresponding series of abscissæ which are in geometrical progression, as was shewn by Alf. Ant. de Sarasa in a tract published (Antv. 1649) in vindication of Greg. de St. Vincent against some aspersions of Marinus Mersennus. Logarithms may also be represented by the "residual arcs" of a parabola (Booth's *New Geometrical Methods*, vol. I. p. 293).

CHAPTER VI.

THE EQUILATERAL HYPERBOLA.

61. The *Equilateral Hyperbola* is a hyperbola whose latus rectum is equal to its axis or latus transversum;* it is also called *Rectangular* since its asymptotes are at right angles. This curve and the circle, which is an equilateral ellipse, may be together designated the *Equilateral Conics.*

The properties of the equilateral hyperbola may for the most part be derived from those of the general hyperbola by equating its axes to one another and to the latus rectum, or by supposing the angle between its asymptotes to become a right angle; but since several of the special results thus obtainable may also be proved independently with peculiar ease, some of them in terms equally applicable to the circle also, we shall here treat the hyperbola in question to a great extent *ab initio*, leaving it to be shewn in the sequel how certain of the properties of the equilateral conics may be transferred to central conics in general by the method of Orthogonal Projection.

62. The latus rectum being supposed equal to the axis, it follows from Art. 33 that

$$PN^2 = AN.A'N = CN^2 - CA^2,$$

which will however be proved independently in Art. 63.

The axes being equal, the radius of the director circle vanishes (Art. 40), or the equilateral hyperbola has no tangents at right angles except its asymptotes. Again, it follows from Art. 45 that every diameter is equal to its conjugate, which leads to many further simplifications; but in this chapter we shall commence by *defining* any diameter which does not meet the curve as equal in length to its conjugate, in pursuance of the analogy between the equilateral hyperbola and the circle or equilateral ellipse.

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* In other words, this hyperbola is called equilateral because *the sides of the* FIGURE *upon its axis* (Schol. A, p. 82) *are equal.*

It is to be noted at the outset (Arts. 35, 54) that *the eccentricity of the equilateral hyperbola is the ratio of the diagonal to the side of a* SQUARE, that the foot X of the directrix bisects CS, and that

$$\tfrac{1}{2}CS^2 = CA^2 = 2CX^2 = 2SX^2.$$

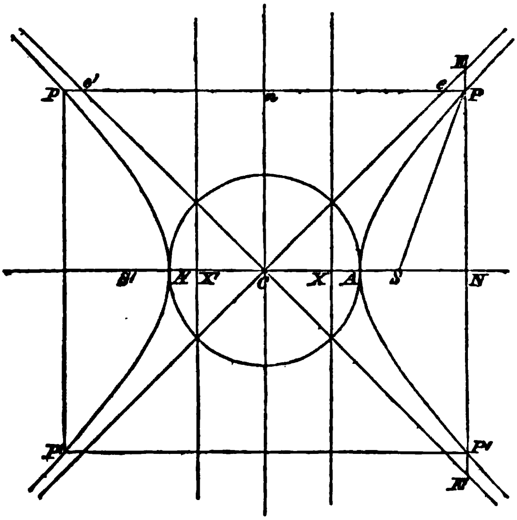
PROPOSITION I.

63. *The principal ordinate of any point on an equilateral hyperbola is a mean proportional to its abscisses.*

If X be the foot of the S-directrix and therefore the middle point of CS, and if PN be the principal ordinate of any point P on the curve, then

$$PN^2 + SN^2 = SP^2 = 2NX^2,$$

and $$CN^2 + SN^2 = 2CX^2 + 2NX^2;$$



therefore $$CN^2 - PN^2 = 2CX^2 = CA^2,$$

or PN^2 is equal to $CN^2 - CA^2$ or $AN \,.\, A'N$.

It is further evident from the figure that if Pn be an ordinate to the conjugate axis,

$$Pn^2 = Cn^2 + CA^2,$$

as might also have been inferred from the consideration that the square of the conjugate semi-axis is $-CA^2$.

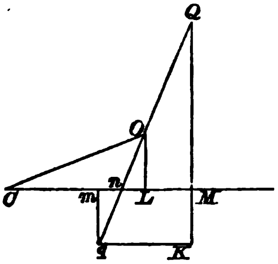
PROPOSITION II.

64. *Any two conjugate diameters of an equilateral hyperbola make complementary angles with either axis and make equal angles with either asymptote.*

(i) If V be any point on the directrix and SZ be at right angles to SV, it is evident that CV and SV are equally inclined to the axis and CV and SZ make complementary angles with the axis. The proposition then follows at once as a special case of Art. 14; it may also be proved independently as below.

(ii) Let Q and q be any two points on the curve, QM and qm their principal ordinates, O the middle point of Qq and OL its ordinate, qK a parallel to the axis meeting QM in K, and n the point in which Qq meets the axis.

Then since $QM^2 + CA^2$ is equal to CM^2 and $qm^2 + CA^2$ to Cm^2,



therefore $$QM^2 \sim qm^2 = CM^2 \sim Cm^2,$$

or $$QM \sim qm : CM + Cm = CM \sim Cm : QM + qm;$$

that is to say,

$$2OL : 2CL = qK : QK,$$

or the angle OCL is equal to the angle qQK.

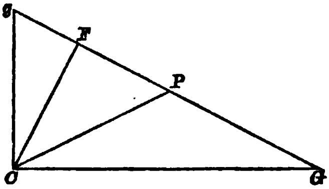
Hence if the chord Qq be parallel to a fixed diameter, the locus of its middle point O will be a second fixed diameter, and the inclinations of the two diameters to either axis will be complementary, and their inclinations to either asymptote will therefore be equal, and conversely.

Corollary 1.

It is evident that any two diameters which are either conjugate or at right angles must lie on opposite sides of an asymptote, and therefore that one of the two and one only meets the curve. It is likewise evident that if two equal diameters be taken on opposite sides of either axis, the one will be equal and at right angles to the conjugate to the other, and conversely that *any two diameters at right angles are equal.*

Corollary 2.

If the normal at P meet the axes in G and g and the conjugate diameter in F, it is evident that PCG is an isosceles triangle having each of its angles at C and G complementary to FCG, and hence that $PG = PC = Pg$, or P is in this case the centre of the circle of Art. 59. Hence or by Art. 45, Cor. 1 the normal is also equal to CD.



Corollary 3.

The angles between any two diameters or chords are equal to the angles between the diameters conjugate thereto. For example, if PQ and PQ' be any two chords drawn from the same point P on the curve and $P'Q$ and $P'Q'$ be the chords supplemental to the former from the further extremity P' of the diameter through P, the angles between the former will be equal to angles between the latter (Art. 44), or *any chord* QQ' *of an equilateral hyperbola subtends at the extremities of any diameter* PP' *angles which are either equal or supplementary.* It will be seen that the angles subtended are *supplementary* when the diameter and the chord intersect WITHIN the curve (as in the figure of Prop. IV.) and *equal* when they intersect WITHOUT the curve.

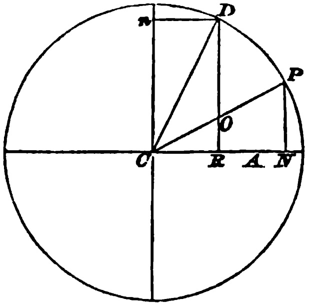
Corollary 4.

The locus of the centre of an equilateral hyperbola circumscribing a given triangle is its nine-point circle, since the diameters to the middle points of its sides contain two and two the same constant angles as the sides to which they are conjugate.* More generally it may be shewn that *the circumscribed circle of any self-conjugate triangle passes through the centre*, since the diameters to its angular points are conjugate to the directions of its opposite sides.

PROPOSITION III.

65. *The projections of any two conjugate semi-diameters upon the axes are alternately equal to one another, and the triangle of which they are adjacent sides is of constant area.*

(i) If CP and CD be conjugate semi-diameters, and PN and DR be principal ordinates and Dn an ordinate to the



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* Since each side and its perpendicular constitute a hyperbola (Art. 54), their intersections belong to the above locus: hence a fresh proof that the feet of the perpendiculars lie on the circle which bisects the sides.

conjugate axis, we have to shew that Cn or DR is equal to CN and that CR is equal to PN; which follows at once from the equality of the radii CP, CD and of the angles PCN, DCn (Prop. II.).

Hence also by Prop. I.,

$$CN^2 - CR^2 = DR^2 - PN^2 = CN^2 - PN^2 = CA^2.$$

(ii) The triangles DCR and PCN being equal, therefore

$$\triangle (DCR + DOP - COR) = PCN + DOP - COR,$$

or

$$\triangle PCD = DRNP = \tfrac{1}{2}(CN + CR)(CN - CR)$$
$$= \tfrac{1}{2}CA^2;$$

which is an equivalent of the theorem that the conjugate circumscribing parallelogram of an equilateral hyperbola is equal to $4CA^2$ (Art. 46).

PROPOSITION IV.

66. *The base of a triangle and the sum or difference of its base angles being given, the locus of its vertex is an equilateral conic.*[*]

(i) If the base and the sum of the base angles of a triangle be given, the third angle is constant and the locus of the vertex is a circle.

(ii) Let $P'CP$ be a fixed diameter of an equilateral hyperbola, V any point in CP produced, and Q any point on the curve. Then since QP and QP' are supplemental chords, the sum of the acute angles which these make with the axis is equal to a right angle (Prop. II), and the sum of their inclinations to the fixed diameter PP' is therefore constant.

The latter constant is at once seen, viz. by removing Q to infinity, to be equal to *twice the angle which the nearer asymptote makes with PP'*.

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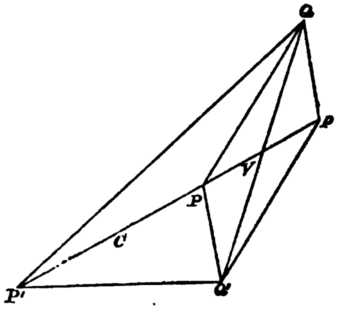
[*] This proposition forms prob. 35 of the *Arithmetica Universalis*, and was suggested by Eucl. III. 21, as is shewn by the preamble: "Ubi angulus ad verticem, sive (quod perinde est) ubi summa angulorum ad basem datur, docuit Euclides locum verticis esse circumferentiam circuli; proposuimus igitur inventionem loci ubi differentia angulorum ad basem datur." NEWTON also stated the corollary given above in the text for the case in which the subtended angles are *equal*.



It follows that the angle at P or its supplement, in the triangle QPP', exceeds the angle at P' or its supplement by a constant quantity; and conversely, that if the base PP' of a triangle be given and the angle at P or its supplement exceed the angle at P' or its supplement by a constant quantity, the locus of the vertex Q of the triangle will be an equilateral hyperbola whereof PP' is a diameter, as was to be proved.

Corollary.

Hence it may be deduced that the angles which any chord of an equilateral hyperbola subtends at the extremities of any diameter are either equal or supplementary, as was shewn independently in Art. 64, Cor. 3.

PROPOSITION V.

67. *At any point of an equilateral hyperbola the ordinate to any diameter which meets the curve is a mean proportional to the abscisses on that diameter.*

Let QV be the ordinate of any point Q on the curve to the diameter PP'; then since the directions of PV and QV and likewise the directions of PQ and $P'Q$ are conjugate, the angle PQV is equal to $QP'V$ (Art. 64, Cor. 3), and the triangles PQV and $QP'V$ are similar, so that

$$PV : QV = QV : P'V;$$

therefore QV^2 is equal to $PV.P'V$ or $CV^2 - CP^2$.

By changing the sign of CP^2 we obtain the corresponding property, viz.

$$QV^2 = CV^2 + CP^2,$$

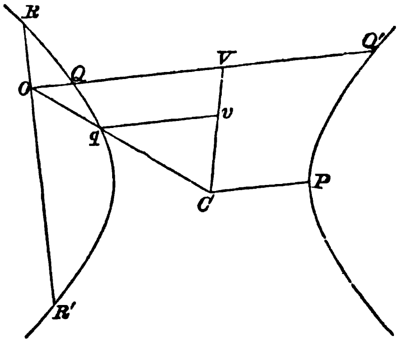
of a diameter which does not meet the curve in real points.

PROPOSITION VI.

68. *The product of the segments of any chord drawn through a fixed point to an equilateral hyperbola varies as the square of the parallel radius.*

Let QQ' be any chord drawn through a fixed point O and V the middle point of QQ', and let CP be the semi-diameter parallel to the chord, q the point in which CO meets the curve or the conjugate rectangular hyperbola, and qv the ordinate of q to the diameter CV.

Then by Prop. V., taking for example the case in which CP and Cq are terminated by the curve,



$$QV^2 - CP^2 : qv^2 - CP^2 = CV^2 : Cv^2 = OV^2 : qv^2.$$

Hence $OV^2 - QV^2 + CP^2 : CP^2 = OV^2 : qv^2 = CO^2 : Cq^2$,

which is a *constant* ratio since O and q are fixed points.

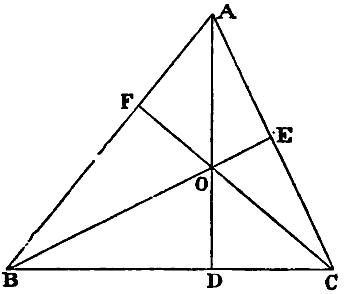
It follows that $OV^2 - QV^2$ or $OQ . OQ'$ varies as CP^2, and if ROR' be any second position of the chord and CP' the radius parallel thereto,

$$OQ . OQ' : OR . OR' = CP^2 : CP'^2.$$

PROPOSITION VII.

69. *If an equilateral hyperbola circumscribes a triangle it passes through its orthocentre, and conversely;* and every conic which passes through the four points of concourse of two equilateral hyperbolas is itself an equilateral hyperbola.*

If ABC be any triangle and AD one of its perpendiculars, any equilateral hyperbola which circumscribes the triangle will have its diameters parallel to AD and BC equal to one another.



The hyperbola therefore meets AD again in a point O such that

$$AD.DO = BD.DC;$$

that is to say, it passes either through the orthocentre of the triangle or through the point in which AD produced meets its circumscribing circle.

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* This theorem was derived from Pascal's hexagram in a memoir by MM. BRIANCHON et PONCELET contributed to Gergonne's *Annales* (tome XI. pp. 205—220) at the commencement of the year 1821, under the title: *Recherches sur la détermination d'une Hyperbole Equilatère, au moyen de quatre conditions données*—viz. thus. Let a hyperbola be described through A, B, C, and the orthocentre O, and let E and F be the two points at infinity on the curve; let H, I, K denote the three points of concourse (AB, OE), (EF, CB), (AF, CO); then HIK is a straight line *parallel to* BC (since I is at infinity) or perpendicular to AO, whence it readily follows that H is the orthocentre of the triangle AOK and that OE the direction of one asymptote is at right angles to AK or OF the direction of the other. The remainder of Prop. VII. follows independently from the fact that by adding together two equations of the form $a(x^2 - y^2) + bx + cy + d = 0$ we arrive at a third equation of the same form: the property of the orthocentre of any triangle is a *special case* of this latter theorem (Art. 54). See Prof. Cayley's *Note on the Rectangular Hyperbola* in the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, vol. I. p. 77 (1862).

But it cannot pass through the latter point, for if so AD and BC would be equally inclined to its axes (Art. 16, Cor. 2) and parallel to its asymptotes, and either B or C would be at infinity: it therefore passes through the orthocentre.

Conversely, any conic which passes through the three angular points and the orthocentre of a triangle must be an equilateral hyperbola.

Moreover, if three of the points of intersection of any two equilateral hyperbolas be taken as the vertices of a triangle, both curves will pass through its orthocentre; and therefore every conic through their four points of concourse must likewise be an equilateral hyperbola.

From this proposition it is manifest that when three points of an equilateral hyperbola are given a fourth can be found; and hence that when four points are given the curve is in general determined.

Corollary.

If BAC be a right angle, the points A and O coalesce and AD touches the curve at A. Hence the tangent at any point A on an equilateral hyperbola may be determined by drawing any two chords AB and AC at right angles and drawing AD perpendicular to BC. If A be a fixed point, BC is constantly parallel to the normal thereat.*

PROPOSITION VIII.

70. *The product of the distances from the centre at which any tangent and the ordinate of its point of contact to any diameter meet the same is constant; and the product of the intercepts on any tangent between the curve and any two conjugate diameters is equal to the square of the parallel radius.*

(i) Let the tangent at Q meet any diameter CP in T, and let QV be an ordinate to that diameter.

Then since CP and CQ are conjugate to QV and QT respectively, they contain equal angles (Art. 64, Cor. 3), so that

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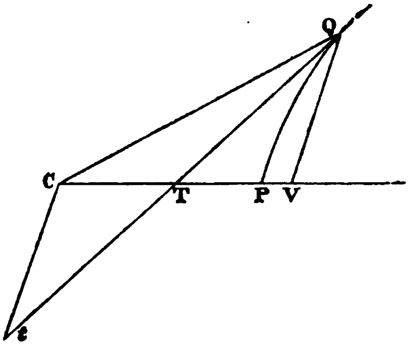
* The fixed point on the normal (Ex. 279) through which BC passes is otherwise seen to be at infinity since when AB and AC are parallel to the asymptotes BC becomes the straight line at infinity.



$$CV : QV = QV : VT.$$

Therefore $\quad CV.CT = CV^2 \sim CV.VT = CV^2 \sim QV^2$

$$= CP^2.$$

(ii) If the tangent at Q makes intercepts CT and Ct on any two conjugate diameters, it may be shewn in like manner that

$$QT : CQ = CQ : Qt,$$

or $\quad QT.Qt = CQ^2 = CD^2,$

where CD is the radius conjugate to CQ.

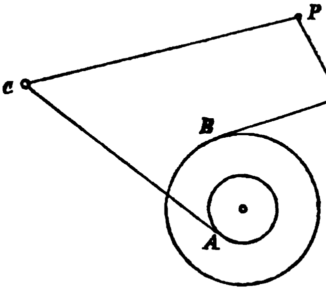
SCHOLIUM.

An excellent machine for describing any number of RECTANGULAR HYPERBOLAS having the same asymptotes was constructed by Mr. H. H. S. Cunynghame, of St. John's College (1873), on the following principle. Let a fixed straight line meet the axis of a rectangular hyperbola at right angles in H; from any point P on the curve draw PM and PN perpendicular to the fixed line and the axis; and on CH produced take HO equal to CA. Then

$$OM + PM = CN + PM = CH;$$

and conversely if O be a fixed point and MP a variable perpendicular to the fixed line HM, then provided that the length OMP is constant the point P will describe a rectangular hyperbola, and its centre C, which is determined by taking HC equal to OMP, will be *independent of the distance* OH. The machine itself consists of a fixed bar HM and a sliding cross bar placed in a horizontal plane: a string fixed at O is kept stretched by a weight in the direction OMP: and a pencil attached at a point P to the string traces an arc of a rectangular hyperbola by the motion of the cross bar. By varying the length OH any number of rectangular hyperbolas having the same asymptotes can be traced with the same length of string.

In a note *On the Mechanical Description of the Cartesian*, by J. Hammond, *Bath, England* (*American Journal of Mathematics, pure and applied*, vol. I. no. 3, p. 283, 1878), the following, applicable to the HYPERBOLA, is given. Suppose two thin circular discs A and B rigidly attached to each other to rotate about their common centre, and suppose the opposite ends of a fine string (which passes through small rings at C and D and is kept stretched by the point of a pencil at P) to be unwound from the two discs. Then will the increments of the lengths CP and DP be as the radii a and b of the discs, and P will describe a curve having the property



$$a \,.\, DP - b \,.\, CP = \text{a constant},$$

which becomes a hyperbola when the discs are equal. If one end of the string be wound on to its disc whilst the other is unwound the curve traced will have the property

$$a \,.\, DP + b \,.\, CP = \text{a constant},$$

and will become an ellipse when the discs are equal.

The mechanical description of the ellipse by the property of Ex. 219 was effected by Guido Ubaldi,[*] who was considered to have made an important discovery; but the property is mentioned by Proclus (on Eucl. I. def. 4) as was remarked in the first volume of the *Ærarium Philosophiæ Mathematicæ*, auctore Mario Bettino, Lib. I. pp. 38—45 (Bononiæ, 1648).

EXAMPLES.

461. Trace the locus of the middle point of a straight line which cuts off a constant area from a corner of a square.

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[*] Guidiubaldi *Planisphaeriorum Universalium Theorica*, Lib. II, end (Pisauri, 1579).

462. Place in a rectangular hyperbola a chord which shall be equal to and be bisected by a diameter of given length.

463. The chords connecting the ends of a fixed diameter of a circle and of any double ordinate of the same intersect upon an equilateral hyperbola.

464. In the rectangular hyperbola the diameter conjugate to the normal at any point is at right angles to the diameter through the point; any two diameters at right angles bisect chords at right angles, and conversely; and any chord subtends equal or supplementary angles at the extremities of a perpendicular chord.

465. The centre of an equilateral hyperbola circumscribing an equilateral triangle is upon the inscribed circle of the triangle, and the centre of the circle is on the hyperbola.

466. The tangents drawn from opposite foci of a hyperbola to any circle which touches both branches intersect upon one of two rectangular hyperbolas, each of them having one asymptote in common with the original hyperbola and having the line joining the foci of the latter for a diameter; and these two rectangular hyperbolas will coalesce if the original hyperbola be rectangular.

467. If two points P and Q move with equal velocities along the arms AB and BC of a right angle, the one starting from A and the other simultaneously from B, and if AA' be drawn equal to AB and in the direction opposite to BQ, shew that $A'P$ and AQ intersect upon a branch of a rectangular hyperbola, and determine its centre and asymptotes.

468. The circles described upon the six common chords of any two rectangular hyperbolas as diameters cut one another orthogonally in opposite pairs.

469. If a parallel to either asymptote of a rectangular hyperbola meet any principal double ordinate PQ in O and the curve in R, shew that

$$\triangle OCR = \tfrac{1}{4} OP.OQ.$$

[illegible] right angles or conjugate in direction [illegible] is a chord of a [illegible] failure of the proof of [illegible] equality of diameters which [illegible] equilateral hyperbola;* [illegible] its asymptotes [illegible] which are parallel to two [illegible]

[illegible] any two diameters [illegible] rectangular hyperbola, shew that [illegible] points in which it [illegible] of the latter; [illegible] from any point of the [illegible] diameters of the latter.

[illegible] rectangular hyperbola, and [illegible] the triangles [illegible]

[illegible] of any triangle [illegible] equilateral hyperbola [illegible] same result [illegible]

[illegible] equilateral hyperbola and the [illegible] determine [illegible]

[illegible] the foci of an [illegible] asymptotes at points [illegible] described [illegible] as to pass

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[illegible] ABC be [illegible] hyperbola, and if [illegible] and BC

[illegible] pole of the [illegible] be the through the vertices meets the curve again at the extremities of one of its own diameters.

476. If PQ and $P'Q$ be any pair of supplemental chords of a rectangular hyperbola, and if the tangent at Q and its ordinate to PP' meet that diameter in T and V, shew that the bisectors of the angle PQP' are parallel to the asymptotes, the segments CP and TP' subtend equal angles at Q, and the circle around CQT touches QV. Shew also that any chord subtends supplementary angles at its pole and the centre of the curve, and that the inclinations of any two tangents to their chord of contact are equal or supplementary to the angles which they subtend at the centre.

477. If a conic pass through the centres of the four circles which touch the sides of a triangle it must be a rectangular hyperbola, and its centre will lie on the circumscribed circle of the triangle.

478. The foci of all the ellipses which can be inscribed in a given parallelogram lie on a rectangular hyperbola passing through its four vertices.

479. The lines connecting the extremities of any two chords drawn through a focus parallel to conjugate diameters of an equilateral hyperbola pass through fixed points on the asymptotes. Examine the cases in which the focal chords coalesce or are parallel to the axis.

480. The axis of the rectangular hyperbola which touches an ellipse and has its axes for asymptotes is a mean proportional to the axes of the ellipse.

481. Construct a rectangular hyperbola having given the centre and a tangent and a point on the curve, or having given an asymptote and a tangent and its point of contact, or having given a diameter and one other point on the curve.

482. The common tangents to the circles described on any two parallel chords of opposite branches of a rectangular hyperbola as diameters subtend right angles at the extremities of the diameter which bisects the chords.

470. Of two chords at right angles or conjugate in direction in an equilateral hyperbola one and one only is a chord of a single branch. Explain the apparent failure of the proof of Art. 16, Cor. 2 which arises from the equality of diameters which are conjugate or at right angles in the equilateral hyperbola;[*] and shew that no circle can intersect the curve or its asymptotes at the extremities of a pair of chords which are parallel to two such diameters.

471. The foci of an ellipse being situated at any two diametrically opposite points of a rectangular hyperbola, shew that the tangents and normals to the ellipse at the points in which it meets the hyperbola are parallel to the asymptotes of the latter; and shew that the tangents to the ellipse from any point of the hyperbola are parallel to conjugate diameters of the latter.

472. If CA be a semi-axis of a rectangular hyperbola, and a perpendicular CY be drawn to the tangent at P, the triangles ACP and ACY will be similar.

473. Prove that the feet of the perpendiculars of any triangle are a conjugate triad with respect to any equilateral hyperbola which circumscribes the triangle; and shew that the same result may also be deduced from Example 76.

474. Given a chord of an equilateral hyperbola and the polar of a given point on the chord, shew how to determine two other points on the curve.[†]

475. The circle described on the line joining the foci of an equilateral hyperbola as diameter meets the asymptotes at points lying upon the tangents at the vertices; and the circle described about any point on the conjugate axis as centre so as to pass

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[*] It is only in accordance with a convention which is not strictly accurate that such diameters are said to be equal. See Chap. IV, Scholium C, p. 101. If ABC be a triangle simultaneously inscribed in a circle and an equilateral hyperbola, and if the perpendicular from A to BC meet the circle in D, the hyperbola in E, and BC itself in F, then $FB \cdot FC = FA \cdot FD = -FA \cdot FE$.

[†] On the given chord AB as diameter describe a circle cutting the polar of the given point O in X and Y; then will the points (AX, BY) and (AY, BX) be the extremities of the chord through O at right angles to AB.

through the vertices meets the curve again at the extremities of one of its own diameters.

476. If PQ and $P'Q$ be any pair of supplemental chords of a rectangular hyperbola, and if the tangent at Q and its ordinate to PP' meet that diameter in T and V, shew that the bisectors of the angle PQP' are parallel to the asymptotes, the segments CP and TP' subtend equal angles at Q, and the circle around CQT touches QV. Shew also that any chord subtends supplementary angles at its pole and the centre of the curve, and that the inclinations of any two tangents to their chord of contact are equal or supplementary to the angles which they subtend at the centre.

477. If a conic pass through the centres of the four circles which touch the sides of a triangle it must be a rectangular hyperbola, and its centre will lie on the circumscribed circle of the triangle.

478. The foci of all the ellipses which can be inscribed in a given parallelogram lie on a rectangular hyperbola passing through its four vertices.

479. The lines connecting the extremities of any two chords drawn through a focus parallel to conjugate diameters of an equilateral hyperbola pass through fixed points on the asymptotes. Examine the cases in which the focal chords coalesce or are parallel to the axis.

480. The axis of the rectangular hyperbola which touches an ellipse and has its axes for asymptotes is a mean proportional to the axes of the ellipse.

481. Construct a rectangular hyperbola having given the centre and a tangent and a point on the curve, or having given an asymptote and a tangent and its point of contact, or having given a diameter and one other point on the curve.

482. The common tangents to the circles described on any two parallel chords of opposite branches of a rectangular hyperbola as diameters subtend right angles at the extremities of the diameter which bisects the chords.

483. If two right angles revolve about opposite extremities of a diameter of a rectangular hyperbola so that the point of concourse of two of their arms is always a point on the curve, their other two arms will make equal intercepts on the normal at that point, and will themselves intersect upon the curve.

484. Tangents (or normals) are drawn in a given direction to a series of confocal conics: prove that the points of contact lie on a rectangular hyperbola passing through the foci and having an asymptote in the given direction.

485. The lines connecting the extremities of any chord and any diameter of a rectangular hyperbola intersect in two points which are concyclic with the extremities of the diameter: determine the condition that they may intersect on a *fixed* circle.

486. Find the points on an equilateral hyperbola at which the normal is parallel to a given chord.

487. The locus of the pole of any chord of a parabola which subtends a right angle at the focus is a rectangular hyperbola.

488. The subnormal at any point of an equilateral hyperbola is equal to the central abscissa; the tangent from the foot of the ordinate to the auxiliary circle is equal to the ordinate; the projection of the normal (terminated by either axis) upon either focal vector is equal to the semi-axis; and the intercept made on any tangent by the asymptotes subtends a right angle at the point in which the normal meets either axis.

489. Any two supplemental chords of a rectangular hyperbola form an isosceles triangle with either asymptote, and conversely.

490. Any two conjugate diameters of an equilateral hyperbola contain equal and similar triangles with the ordinates and abscissæ of their extremities to any other diameter.

491. The sum or difference of the inclinations of any two conjugate diameters of an equilateral conic to a fixed diameter is constant: distinguish between the several cases.

492. Any circle drawn through the extremities of a diameter of a rectangular hyperbola meets the curve again at the extremities of a diameter of the circle, and its tangents at those points are ordinates of the diameter of the hyperbola.

493. The circles described on parallel chords of a rectangular hyperbola as diameters have a common radical axis.

494. The ends of the equal conjugate diameters of a series of confocal ellipses lie on the confocal rectangular hyperbola.

495. The straight line joining the feet of the perpendiculars from any point of a rectangular hyperbola to two conjugate diameters is parallel to the normal at the point.

496. The opposite arcs cut off by any two diameters of a rectangular hyperbola subtend equal angles at any point on the curve.

497. Any two rectangular hyperbolas so placed that the axes of the one coincide with the asymptotes of the other intersect at right angles, and each of their common tangents subtends a right angle at the centre; and if two tangents to a pair of conjugate rectangular hyperbolas be at right angles, the straight line joining their points of contact subtends a right angle at the centre.

498. If on opposite sides of any chord of a rectangular hyperbola equal segments of circles be described, the four points in which the completed circles meet the hyperbola again will be the angular points of a parallelogram; and if parallels be drawn from any point on a rectangular hyperbola to the sides of an inscribed parallelogram, they will meet its opposite sides in two pairs of points lying on a circle.

499. The foot of the focal perpendicular upon any chord of a rectangular hyperbola which subtends a right angle at the focus lies on a fixed straight line.

500. The normal at any point P of a rectangular hyperbola meets the curve again in Q, and RR' is a chord parallel to the

normal: prove that PR, QR' and PR', QR intersect on the diameter at right angles to CP.[*]

501. In any right angled triangle inscribed in an equilateral hyperbola the perpendicular upon the hypotenuse is the tangent at the right angle. Hence shew how to find a third point on the curve when two points and the tangent at one of them are given; and shew that the curve is determined when two points and the tangents thereat are given, or three points and the tangent at one of them, or two points and the tangent at one of them and a second tangent.

502. Given the middle points and the directions of two chords of an equilateral hyperbola, the two points and the intersection of the parallels through each point to the opposite chord determine a circle which passes through the centre of the hyperbola.

503. If through each of two points a parallel be drawn to the polar of the other with respect to an equilateral hyperbola, the circle through the two points and the intersection of the parallels will pass through the centre of the hyperbola.

504. Given the centre of a rectangular hyperbola and a self-conjugate triad, determine its asymptotes.[†]

505. Two equilateral hyperbolas can be inscribed in a given quadrilateral, and their centres are at the points in which the diameter[‡] of the quadrilateral meets the circumscribed circle of the triangle formed by its three diagonals.

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[*] Examples 471, 482, 484, 492—500 and others are from Wolstenholme's *Mathematical Problems*.

[†] If C be the centre and PQR the conjugate triad, let CP meet QR in V, and upon QR take points Q' and R' such that $Q'V = R'V = CV$; then will CQ' and CR' be the asymptotes. The following method applies to the *general* hyperbola (Ex. 455). Draw CP', CQ', CR' parallel to QR, RP, PQ, and find the two double lines of the involution determined by the pairs of conjugate rays CP, CP'; CQ, CQ'; CR, CR'.

[‡] By a theorem of NEWTON (*Principia*, Lib. I. sect. V. lemma 25, cor. 3) the centres of all the conics inscribed in a quadrilateral lie upon the straight line (Ex. 372) which we have called the DIAMETER of the quadrilateral (p. 138). See also Ex. 513.

506. The three pairs of chords connecting any four points on an equilateral hyperbola intersect upon the circumference of a circle which passes through its centre.

507. The nine-point circles of the four triangles determined by four given points cointersect at the centre of the equilateral hyperbola which passes through the four points.[*]

508. Four points being taken at random in a plane, there exists in general one other point in the same and one only such that the lines radiating therefrom to the middle points of the six lines connecting the four points are inclined at the same angles as the lines which they severally bisect.

509. Given any two points in the plane of an equilateral hyperbola and the directions conjugate to the diameters passing through them, determine a circle on which the centre lies. If a chord and the direction of the polar of a point upon it be given, this circle passes through the point and bisects the chord and has its tangent at the middle point of the chord in the given direction.

510. Determine the centres of the four equilateral hyperbolas which pass through two given points and touch two given lines.[†]

511. Given two points of an equilateral hyperbola and two tangents to the same, determine the four positions of the chord of contact.[‡]

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[*] Three other circles may be determined by Ex. 502 and another by Ex. 506, making in all EIGHT, which pass through the same point.

[†] If A and A' be the given points, C the intersection of the tangents, and X and Y the points in which they meet AA', the points A, A' and X, Y determine an involution through one of whose foci P or Q the chord of contact of the two tangents must pass: let it pass through P, of which CQ will be the *Polar*. Bisect AA' in I and XY in C, and draw a circle through P and I having its tangent at I parallel to CQ (Ex. 509). Through the second intersection of CI with the circle draw Px meeting CK in x, and draw the tangent to the circle and let it meet CQ in y: then the intersections of xy with the circle determine two of the required centres, and the other two are determined by interchanging P and Q. This construction is given by PONCELET in Gergonne's *Annales*, tome XII., where he corrects (p. 244) the misstatement of the joint article by Brianchon and Poncelet (XI. 218) that *the four centres lie on* ONE *circle*.

[‡] Determine as before the point P on the chord of contact and its polar CQ; find a third point D on the curve (Ex. 474); and let F and F' be the foci of the involution determined by A, D and the pair of points in which the tangents meet AD. Then will PF and PF' be two positions of the chord of contact.

512. Given that the centres of all the equilateral hyperbolas circumscribing a triangle lie on a circle, deduce the fundamental property of the nine-point circle of any triangle.

513. The three circles whose diameters are the diagonals of any quadrilateral belong to a coaxal system,* whose limiting points are the centres of the two inscribed equilateral hyperbolas.

514. The director circles of all the central conics touching the same four lines have a common radical axis,† which is also the directrix of the inscribed parabola; and if the conics touch but three lines, their director circles have a common radical centre.

515. The circumscribed circle of any triangle which is self conjugate with respect to a conic cuts its director circle orthogonally:‡ if the conic be an equilateral hyperbola the circle passes through its centre: if it be a parabola its directrix passes through the centre of the circle.

516. The base of an isosceles triangle being upon a fixed straight line and each of its equal sides passing through a fixed point, whereof one is on the fixed line, shew that the locus of the vertex of the triangle is an equilateral hyperbola passing through the fixed points and having an asymptote parallel to the fixed line.

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* See TOWNSEND'S *Chapters on the Modern Geometry of the Point, Line, and Circle*, Art. 189 (vol I. p. 253).

† This follows from Prop. VII by reciprocation, as in the *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. I. p. 159. A direct proof by involution is given in vol. III. p. 31 of the same, by "W. K. C." [CLIFFORD.]

‡ It may be shewn that the circumscribed circle of the triangle formed by the three diagonals of a quadrilateral is orthogonal to the circles on its three diagonals as diameters. Ex. 515 then follows with the help of Ex. 514 by regarding the sides of any self conjugate triangle as the diagonals of a quadrilateral which envelopes the conic. This theorem is due to GASKIN, who proved it by the Cartesian method in his work (p. 33) already referred to in the note on Ex. 458. Eight years later the equivalent theorem: "*On donne un triangle conjugué à une ellipse...la tangente menée du centre de l'ellipse au cercle circonscrit au triangle est égale à la corde du quadrant d'ellipse,*" was proposed by Cap. Faure as Quest. 524 in the *Nouvelles Annales*, tome XIX. p. 234 (1860). See also XIX. 290, 345; XX. 25, 77; V. 308 (2^{me} série).

517. If through five concyclic points taken in fours five equilateral hyperbolas be drawn, their centres will lie on a second circle of diameter equal to the radius of the former.

518. The locus of the centres of all the conics which pass through four given points is a conic. Prove that the locus will reduce to a *circle* if any two of the conics through the four points be equilateral hyperbolas, and to an *equilateral hyperbola* if the four points lie on a circle.

519. The angular points and the centroid and orthocentre of any triangle determine ten triangles whose nine-point circles meet in a point; and this point lies on the circumference of the maximum ellipse that can be inscribed in the original triangle.*

520. Shew that the centre of any equilateral hyperbola inscribed in an obtuse angled triangle lies upon the circle with respect to which the triangle is self conjugate.

521. The angular points of a triangle and the extremities of any diameter of its circumscribing circle, taken four together, determine five equilateral hyperbolas whose centres lie on the nine-point circle of the triangle.

522. A variable triangle circumscribes an equilateral hyperbola and its nine-point circle passes through the centre of the curve: prove that the locus of the centre of its circumscribed circle is the hyperbola in question.

523. Prove that the opposite sides AB and CD of a parallelogram inscribed in a rectangular hyperbola subtend either equal or supplementary angles at any point P on the curve; the circumscribed circles of the triangles PAB, PBC, PCD, PDA are equal; and the product of the perpendiculars from P to each pair of opposite sides of the parallelogram is the same.

524. With the extremities of any diameter of the circumscribed circle of a triangle as foci two parabolas are drawn

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* See *Mathematical Questions, &c. from the* EDUCATIONAL TIMES, vol. IV, p. 89.

touching the sides of the triangle; prove that the tangents at their vertices are the asymptotes of one of the series of rectangular hyperbolas which pass through the vertices of the triangle.

525. Given the base of a triangle, prove that if the bisectors of its vertical angle be parallel to fixed lines, or if its two sides make equal angles with two fixed straight lines, the locus of its vertex will be a rectangular hyperbola whose asymptotes bisect the base of the triangle and are parallel to the bisectors of its vertical angle.

526. Given two fixed tangents to a variable parabola and a fixed point on its axis, prove that the locus of its focus is a rectangular hyperbola having its asymptotes parallel to the bisectors of the angle between the fixed tangents and its centre at the bisection of the line joining their point of concourse to the fixed point on the axis.

527. If a rectangular hyperbola has double contact with a parabola, the line joining the intersection of their common tangents with the centre of the hyperbola is bisected by the directrix of the parabola.

528. The circle described with any diameter of an equilateral hyperbola as radius meets the curve again in three points which determine an equilateral triangle; and conversely, the circumscribed circle of any equilateral triangle inscribed in an equilateral hyperbola has one of its radii coincident with a diameter of the hyperbola. If OA and OB be the bounding radii of a circular arc AB, shew that a point of trisection of the arc lies upon the rectangular hyperbola which has OA for a diameter and passes through the point of concourse of OB with the tangent at A to the circle. Deduce from the above that the problem, to trisect a given angle, admits of three solutions. Prove also that that if points P and Q be taken on AB such that

$$\text{arc } AP = 2 \text{ arc } BQ,$$

the intersection of AP and OQ will lie on the hyperbola.

529. A pair of mean proportionals to two given magnitudes m and n may be found as follows. Describe a parabola of latus rectum equal to m, and with its axis and the tangent at its vertex as asymptotes describe a hyperbola whose semi-latus rectum is a mean proportional to m and $2n$; then will the distances of their point of concourse from the asymptotes be the two mean proportionals which were to be found.*

530. The circle described on any radius of a rectangular hyperbola as diameter meets the curve in two points whose distances from the asymptotes are in continued proportion; and conversely, the hyperbola drawn through the point of concourse of two sides of a rectangle so as to have the other two sides for asymptotes meets the circle circumscribing the rectangle in a second point whose distances from the asymptotes are a pair of mean proportionals to the sides of the rectangle. Hence shew how to find a pair of mean proportionals to two given magnitudes.†

531. The difference of the ordinates of the points in which any tangent to an equilateral hyperbola meets the directrices is to the difference of their distances from the centre as the diagonal to the side of a square; and their distances from the centre are to one another as the focal perpendiculars upon the tangent.‡

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* This construction also (cf. Art. 20, Cor.) is ascribed to Menaechmus.

† The *Delian* problem of the DUPLICATION OF THE CUBE (*i.e.* the construction of a cube of twice the volume of a given cube), which so exercised the ancient geometers, was reduced by Hippocrates of Chios to the problem of finding a pair of mean proportionals to two given magnitudes (Art. 20, Cor. and Exx. 432, 529, 530). See Reimer's *Historia Problematis de* CUBI DUPLICATIONE (Gottingæ, 1798); Walton's *Problems in illustration of the principles of Plane Coordinate Geometry*, p. 157; Bretschneider's *Die Geometrie und die Geometer vor* EUKLIDES, §78. The method of Ex. 530 is employed in Grégoire de St Vincent's *Opus Geometricum Quadraturæ Circuli* (Lib. VI. prop. 138, p. 602), and elsewhere. The TRISECTION OF THE ANGLE (Exx. 308, 390, 528) like the former problem is equivalent to the solution of a cubic equation, and either may be effected by the intersection of a circle with a parabola as was proved, in the third book of his *Geometria*, by DES CARTES; who further shews that solid problems in general can be reduced to the same two constructions, and gives his reasons for concluding *a priori* that "*Problemata Solida construi non possint absque Sectionibus Conicis, nec quæ magis composita sunt sine aliis lineis, magis compositis.*"

‡ Exx. 531—7 are from Booth's *New Geometrical Methods*, i. 291—2 and i. 343; Exx. 538—40 from Gregory St. Vincent's *Opus Geom. Quadrat. Circuli*, Lib. VI. props. 146, 156, 166 (pp. 606–16).

532. The auxiliary circle of an equilateral hyperbola is the envelope of the lines joining the points in which any two diameters at right angles meet the curve and its directrices respectively.

533. If tangents be drawn to an equilateral hyperbola from a point on one of its directrices and their chord of contact be produced to meet the directrix, the intercept upon it between the chord and the point will subtend a right angle at the centre; and if the tangents be drawn from any point not on the directrix, the focal distance and the polar of the point will intercept on the directrix a length which subtends a right angle at the centre.

534. The intercepts on either directrix of an equilateral hyperbola between any chord and the tangents at its extremities subtend equal angles at the centre.

535. The chords drawn from any two fixed points on an equilateral hyperbola to a variable point on the same intercept on either directrix a length which subtends a constant angle at the centre, the constant angle being a right angle in the case in which fixed points are the vertices; and the angles subtended at the centre by the intercepts on the two directrices are together equal to the angle subtended by the chord joining the fixed points.

536. If a right angle revolve about the centre of an equilateral hyperbola, the abscissa of any point on either arm varies inversely as the abscissa of the point in which its polar meets the other arm.

537. If a diameter of a parabola meet the curve in P and the directrix in M and a length MPQ be taken on it equal to the normal at P, the locus of Q will be a rectangular hyperbola having its centre at the vertex of the parabola. If $M'P'Q'$ be any second position of MPQ, shew that the hyperbolic area $QMM'Q'$ is equal to the product of the arc PP' of the parabola and its semi-latus rectum.*

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* When the diameters are consecutive the distance between them is to the arc PP' as the subnormal at P to the normal, whence the required result readily follows. Thus the QUADRATURE of the Hyperbola is reduced to the RECTIFICATION of the Parabola.

538. A hyperbola having for asymptotes the axis of a parabola and the tangent at its vertex cuts the parabola in O, and APQ is drawn from the vertex of the parabola to meet it in Q and to cut the hyperbola in P; prove that if the ordinate of Q cut the hyperbola in R, the segment AOP is equal to one-third of the segment APR; and if from the latter segment $AO'P$ be cut off equal to one-third of its area, then will AO' and the ordinate of O' meet QR and AQ respectively on a parallel through O to the axis of the parabola.

539. If from any two points Q and Q' on the above hyperbola parallels be drawn to its asymptotes meeting the curve in M, M' and N, N', the areas OQM, OQN, $OQ'M'$, $OQ'N'$ will be proportionals.

540. If through the point Q a second parabola be drawn having the asymptotes for its axis and the tangent at its vertex, the arcs of the two parabolas will trisect the area QMN.

NOTE ON THE NINE-POINT CIRCLE.

The property of the Nine-point Circle was stated and proved by Brianchon and Poncelet in Gergonne's *Annales*, XI. 215 (1821). See above, p. 175, note. The property may be verified as suggested in Ex. 512, viz. thus. Each of the six chords connecting a triad ABC and its orthocentre O (Art. 69) is a diameter of one of the series of equilateral hyperbolas which can be drawn through A, B, C: these six chords are therefore bisected by the locus of centres (a circle), which also contains the three intersections D, E, F of the chords taken in opposite pairs (Art. 54 and p. 171, note). A short proof by inversion of the theorem (Salmon's *Conic Sections*, Art. 131, Ex.), that *the nine-point circle of a triangle touches its inscribed and exscribed circles*, was given by Mr. J. P. Taylor, Fellow of Clare College, in the *Quarterly Journal of Mathematics*, vol. XIII. p. 197. The same nine-point circle touches the SIXTEEN inscribed and exscribed circles of the four triangles determined by a triad and its orthocentre.

CHAPTER VII.

THE CONE.

71. An unlimited straight line which passes through a fixed point in space and moves round the circumference of a fixed circle generates a surface which is called a *Cone*.[*] The line in any of its positions is called a *Side* or a *Generating Line* of the cone; the fixed point is called the *Vertex*, and the straight line joining it to the centre of the circle is called the *Axis* of the cone.[†]

When the axis is at right angles to the plane of the circle the surface generated is a *Right Circular* cone: in other cases the cone is called *Oblique* or *Scalene*. In this chapter we shall shew that the curve of intersection of a cone with a plane is a parabola, an ellipse, or a hyperbola; and we shall derive their elementary properties from the cone itself, confining our attention in general, for the sake of simplicity, to the right circular cone.

In the particular case in which the section of a right circular cone is taken at right angles to its axis, it is evident that the section is a circle. Any circular section may be regarded as the *Base* of the cone.

The *Focal Spheres* of any plane section of a right circular cone are the spheres which can be inscribed in the cone so as to touch the plane of section. Their points of contact may be defined as the *Foci*, and the intersections of their planes of contact with the plane of the section as the *Directrices* of the section. We shall shew that these points and lines are identical with the foci and directrices as hereinbefore defined.

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[*] The complete cone consists of two infinite portions on opposite sides of the vertex. The (right) cone as defined by EUCLID (Book XI. def. 18) is the finite figure (p. 193) described by the revolution of a right-angled triangle about one of the sides containing the right angle.

[†] The cone and its axis are thus defined by APOLLONIUS at the beginning of his Περὶ Κωνικῶν (p. 13, ed. Halley). In the oblique cone, which has two sets of circular sections, this definition gives two lines, either of which may be called the "axis." In analytical treatises on Solid Geometry the term axis is *not* used as above.

In what follows suppose a plane through the axis and at right angles to the base of the cone to be taken as the *Plane of Reference* and the *Section* to be made by a plane at right angles thereto.

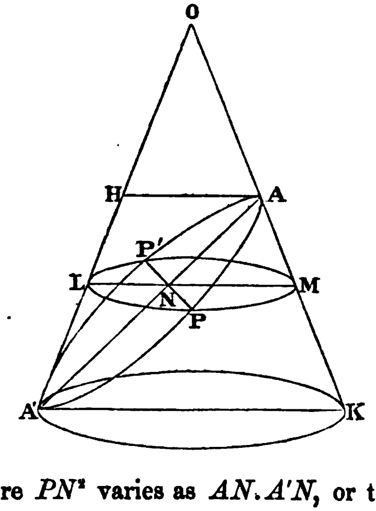
THE ORDINATE.

PROPOSITION I.

72. *The square of the principal ordinate in any section varies as the product of the corresponding abscissæ.*

(i) Let A and A' be the vertices of the section, PP' a principal double ordinate meeting AA' in N, and let the plane of circular section through PP' meet OA' in L, and OA in M, the point O being the vertex of the cone.

Then in the circle PN^2 is equal to $LN.MN$. And as LM moves parallel to itself, MN varies as AN and LN varies as $A'N$. Therefore PN^2 varies as $AN.A'N$, or the square of the ordinate varies as the product of the abscisses.



When the section cuts all the generating lines on the same side of the vertex it is an *Ellipse*, and when it cuts both branches of the cone (fig. p. 199) it is a *Hyperbola.*

(ii) If the axis AN of the section be *parallel* to the side OL of the cone, then, in the figure of Art. 74, since the length LN is constant whilst NR varies as the abscissa AN, therefore PN^2 (or $LN.NR$) varies as AN, and the section is a *Parabola*.

Hence it appears that whatever be the vertical angle of the cone the section is a parabola, a hyperbola or an ellipse according as the angles LOA and OAN are together equal to or greater or less than two right angles.

Corollary 1.

Since in the former part of the proposition

$$PN^2 : AN.A'N = LN.MN : AN.A'N = AH.A'K : AA'^2,$$

where AH and $A'K$ are the diameters of circular sections, it follows that *the conjugate axis of the section is a mean proportional to the diameters of the circular sections through its vertices*, and the semi-axis conjugate is a mean proportional to their radii or to the perpendiculars from the vertices of the section upon the axis of the cone.

Corollary 2.

Hence it readily follows that the orthogonal projection of the section upon a plane of circular section is a conic having a focus upon the axis of the cone.*

SCHOLIUM A.

MENÆCHMUS (or Menechmus) is said to have been the discoverer of the conic sections, which have been accordingly called after him the *Menæchmian Triads*. Thus Proclus in the second book of his commentaries on the First Book of Euclid, writing on Def. 4, states upon the authority of Geminus: "But with respect to these sections, the conic were invented by Mænechmus (*sic.*), which also Erastosthenes relating says, *Nor in a cone Mænechmian* ternaries divide*" (Thos. Taylor's *Proclus*, I. 134); and see the end of the letter of Erastothenes to Ptolemy, given by Eutokius in his commentary on Archimedes, De *Sphær. et Cyl.* (Archim. *Op.*, p. 146, ed. Torelli), where the same verse,

μηδὲ Μενεχμείους κωνοτομεῖν τριάδας,

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* This property was given by W. H. Talbot, of Cambridge, in Gergonne's *Annales*, XIV. 126.

appears in its original context. The authorities are given as above in Bretschneider's, *Die Geometer und die Geometrie vor Euklides*, § 116, p. 155.

The parabola, the ellipse, and the hyperbola were anciently regarded as the sections of right circular cones of different angles by planes *at right angles to their sides*, and were accordingly known as the sections of the right-angled, the acute-angled, and the obtuse-angled cones respectively. Apollonius shewed that they could all be cut from one and the same right or scalene cone, and he gave them their names *Parabola*, *Ellipse*, *Hyperbola*, for the reason assigned above in Chap. IV. Scholium A, p. 82. See Pappi Alex. *Collectio*, lib. VII. § 30 (p. 672, ed. Hultsch); and J. H. T. Müller's *Beiträge zur Terminologie der Griechischen Mathematiker*, p. 25 (Leipzig, 1860). Archimedes is sometimes wrongly supposed to have employed the term *Parabola*, for the reason that one of his treatises came to be known by the title, Ἀρχιμήδους τετραγώνισμος Παραβολῆς, whereas throughout the treatise the author uses only the periphrasis, ἡ τοῦ ὀρθογωνίου κώνου τομά. In like manner he calls the ellipse ἡ τοῦ ὀξυγωνίου κώνου τομά, and the exceptional occurrence of the term *Ellipse* itself in his work *De Coneid. et Sphæroid.* (lib. I. cap. 9, &c.) is rightly attributed to an error of transcription.

Eutokius, at the commencement of his commentary on the *Conics* of Apollonius (p. 9, ed. Halley) explains the names of the three conics as follows.* Let LOR be the angle of the cone, AN the axis of the section supposed *at right angles to the side* OR, and A the vertex of the section, which will be a Parabola, a Hyperbola, or an Ellipse, according as the angle of the cone is equal to, or greater or less than a right angle. The Parabola is accordingly said to be so called because AN is *parallel* to OL: the Hyperbola because the angles LOA and OAN together *exceed* two right angles, or because NA falls beyond the vertex and meets the side LO produced: and the Ellipse because the angles LOA and OAN are together *less than* two right angles, or because it is a defective circle (κύκλον ἐλλειπῆ). If however the names in question were first introduced by Apollonius, it is clear that they are to be explained as on p. 82. The property of the ordinate there given is used by him to discriminate between the three conics and forms the actual basis of his investigations, so that having once obtained it he makes in reality very little further use of the cone.

THE ASYMPTOTES.

PROPOSITION II.

73. *The sections of a cone by parallel planes are similar curves; and the asymptotes of the hyperbolic sections made by*

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* The passage is given in the Greek at the end of Walton's geometrical *Problems* (see above, Ex. 530, note).

parallel planes are parallel to the sides of the cone which lie on the parallel plane through its vertex.

If any fixed straight line through the vertex O of the cone meet a pair of parallel planes in M and N, and if a variable plane through OMN meet the sections made by those planes in P and Q, then

$$MP : NQ = OM : ON;$$

or the parallel vectors MP and NQ are in a constant ratio, and the sections are therefore similar.

If M and N be the centres of a pair of hyperbolic sections the vectors MP and NQ become infinite together: hence the asymptotes of any two parallel hyperbolic sections are parallel to one another, and therefore also to the sides of the cone determined by the parallel plane through the vertex, since this is a limiting position of one of the planes of section.

Corollary.

The angle between the asymptotes of a hyperbolic section cannot exceed the vertical angle of the cone; and conversely in order to cut a hyperbola of given eccentricity from a cone we must take a cone whose vertical angle is not less than that between the asymptotes.

THE FOCAL SPHERES.

PROPOSITION III.

74. *The distance of any point of a section from the point of contact of its plane with either focal sphere is in a constant ratio to the distance of the point from the plane of contact of the sphere with the cone, or to its distance from the line in which that plane meets the plane of section.**

Let S be the point in which the plane of the section touches either of its focal spheres, and *MX* the line in which it meets the plane of contact of the sphere with the cone.

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* The reader who prefers to define a conic as the section of a cone by a plane may define its foci and directrices by means of the focal spheres (p. 192), as Pierce Morton (Schol. B) proposed to do. The proposition will then take the form that "The distance of any point on a conic from either focus is in a constant ratio to its distance from the corresponding directrix."

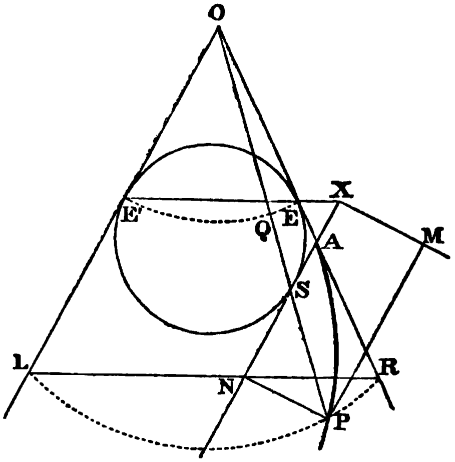
Take any point *P* on the section, and let *Q* be the point in which the side of the cone through *P* touches the sphere, and let *PM* be supposed parallel to the axis of the section.

(i) Then the tangent *PS* to the sphere is equal to the tangent *PQ*, and the perpendicular from *P* to the plane of contact varies as *PQ*, and likewise as *PM*; and therefore *SP* varies as that perpendicular, and likewise as *PM*.*

Hence the point of contact *S* and the line *MX* are a *Focus* and *Directrix* in accordance with their definition on p. 1.

(ii) This result is usually obtained, rather less directly, as follows.

Having made the same construction, let the side of the cone through the vertex *A* of the section touch the sphere in *E*



and meet the plane of circular section through *P* in *R*; let *PN* be the ordinate of *P* to the axis *AN* of the section, and let *X* be supposed to lie in the plane of reference.

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* If α and β be the inclinations of the axis of the cone to the axis of the section and to a side of the cone respectively, then $SP = \cos\alpha . \sec\beta . PM$; or *the eccentricity* is equal $\cos\alpha . \sec\beta$, and is therefore limited by the vertical angle of the cone and *cannot exceed* $\sec\beta$.

Then since SP is equal to PQ, and PQ is equal to RE, therefore SP is equal to RE.

Hence and by parallels,

$$\begin{aligned} SP : NX = RE : NX &= AE : AX \\ &= AS : AX, \end{aligned}$$

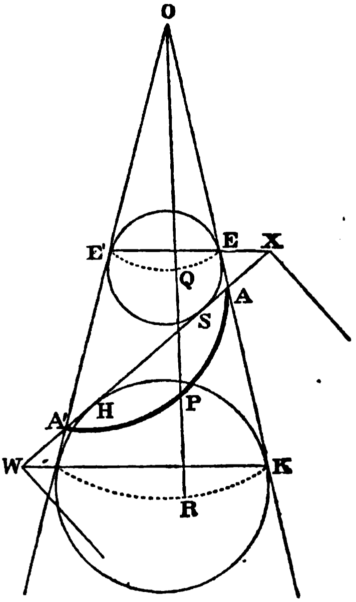
or SP is to PM or NX in the constant ratio of SA to AX.

In the case of a bifocal conic the second focus and directrix are determined in like manner, as is indicated in the diagrams of Art. 75.

PROPOSITION IV.

75. *The sum or difference of the distances of any point on a section from the points of contact of its plane with the focal spheres is constant, being equal to the intercept made by the planes of contact of the spheres upon any side of the cone.*

Let S and H be the foci, or points of contact of the focal spheres, and Q and R the points in which the spheres meet the generating line through any point P of the section.

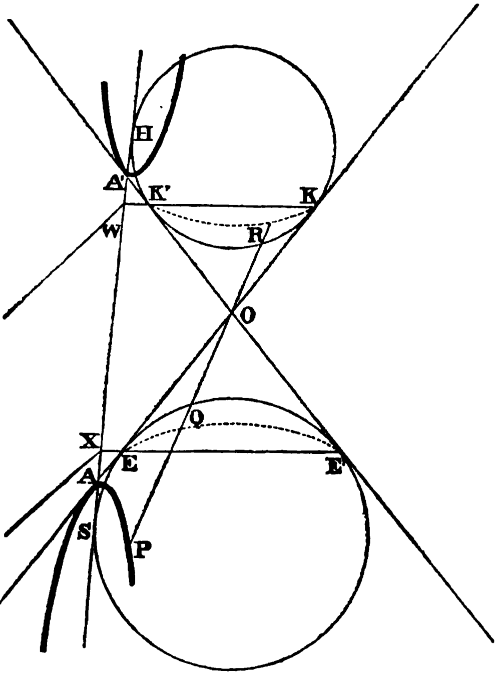


(i) Then since the tangents from P to either sphere are equal, therefore by addition in the case of the ellipse,

$$SP + HP = PQ + PR = QR,$$

which is the same for all positions of P on the section.

(ii) And by subtraction in the case of the hyperbola,



$$SP \sim HP = PQ \sim PR = QR,$$

which is constant, as in the former case.

Corollary.

In the first figure if OA and OA', drawn from the vertex of the cone to the vertices of the section, touch the S-sphere in E and E', then

$$OA' - A'S = OE' = OE = OA - AS,$$

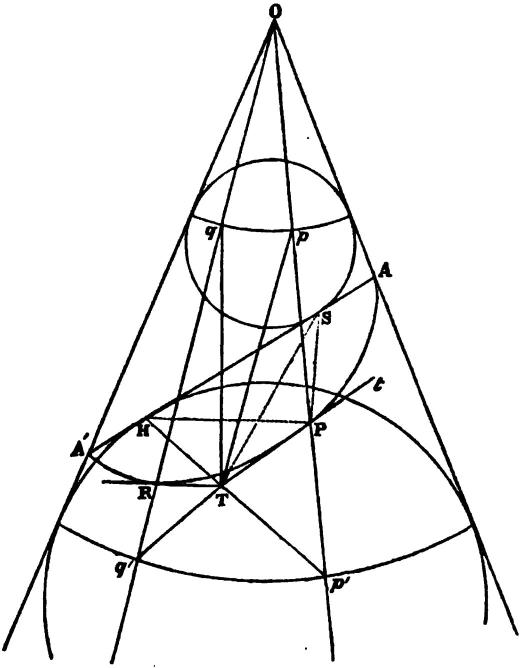
or $OA' \sim OA$ is equal to SH. In the second figure it may be shewn in like manner that $OA' + OA$ is equal to SH. Hence the eccentricity is the ratio of $OA' \pm OA$ to AA', and the distance of the foci from the centre is $\frac{1}{2}(OA' \pm OA)$.

PROPOSITION V.

76. *The tangent at any point of a section makes equal angles with the focal distances and with the side of the cone.*[*]

Let TPt be the tangent at any point P to the section, and let the side OP of the cone meet the focal spheres in p and p'.

Then since the tangents PS and Pp to the S-sphere are equal, and likewise the tangents TS and Tp, therefore the



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[*] This property and its applications were pointed out by me in an article on *An Angle-property of the Right Circular Cone* contributed in June 1871 to the *Messenger of Mathematics* (vol. I. p. 67), and in subsequent articles. The same methods were employed in Booth's *Treatise on Conics* published some years later in the second volume of his *New Geometrical Methods*; but from the introduction to that volume we learn (p. x) that the substance of the treatise had been read before the Royal Irish Academy in 1837, although not published till forty years after.

triangles TSP and TpP are equal in all respects, having their angles at T equal and their angles at P and their supplements equal.

In like manner it may be shewn that the angles at T and those at P in the triangles THP and $Tp'P$ are equal.

Hence $\angle SPt = pPt = p'PT = HPT$,

or the tangent TPt makes equal angles with the focal distances SP and HP and with the side OP of the cone.*

PROPOSITION VI.

77. *If tangents be drawn to a section from any point in its plane, and a side of the cone be drawn through either point of contact, the intercept upon it between the focal spheres subtends at the point of concourse of the tangents an angle equal to the angle between them.*

It may be shewn as in Art. 76 that the angles STP and pTP are equal, where T is any point on the tangent at P; and in like manner that the angles HTP and $p'TP$ are equal.

Hence $\angle pTp' = STP + HTP = STH + 2STP.$

If TR be the second tangent from T to the section, and if the side of the cone through R meet the spheres in q and q', it may be shewn in like manner that

$$\angle qTq' = STR + HTR = STH + 2STR.$$

And since the triangles pTp' and qTq' have their sides which touch the spheres equal and their bases pp' and qq' equal, their angles at T are equal. Hence a fresh proof that the angle STP is equal to HTR (Art. 50); and it follows that

$$\angle PTR = pTp' = qTq',$$

as was to be proved.†

Corollary.

If PTR be a right angle, pTp' is a right angle and T lies on a certain sphere. The locus of T is therefore the section of a sphere by a plane; that is to say, it is the circle which is called the Director Circle.

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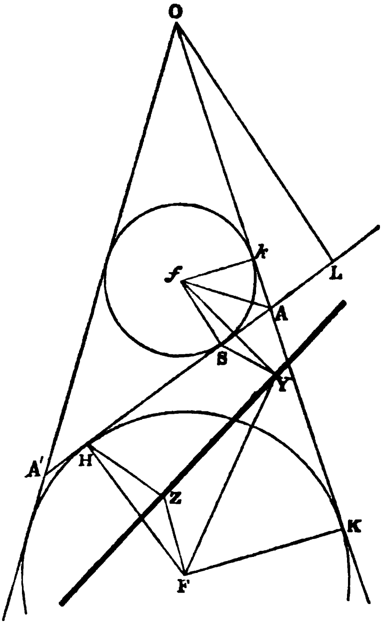
* This may also be proved by the method of Art. 48 (i), since $OP \sim SP$ is constant.

† Observe that the triangle Tpp' is identically equal to the triangle STH' of Art. 50 (Cor. 2 and Scholium D).

PROPOSITION VII.

78. *The conjugate axis of any section is a mean proportional to the diameters of its focal spheres, and its latus rectum varies as the perpendicular to the plane of section from the vertex of the cone.*

Let AA' be the axis of the section and f and F the centres of its focal spheres.



(i) Then since fA and FA bisect the supplementary angles between AA' and the side of the cone through A, therefore by similar triangles fSA, FHA,

$$fS : AS = AH : FH,$$

and therefore $$fS \, . \, FH = AS \, . \, AH = CB^2,$$

or CB is a mean proportional to the radii of the spheres, and $2CB$ to their diameters.

(ii) Draw OL perpendicular to AA', and draw fk and FK to the points of contact of the spheres with the side OA of the cone.

Then since OK is equal to the semi-perimeter of the triangle OAA',

$$fk \,.\, OK = \triangle\, OAA' = \tfrac{1}{2} OL \,.\, AA',$$

where OK varies as the radius FK.

Therefore $OL \,.\, AA'$ varies as $fk \,.\, FK$ or CB^2; that is to say, OL is in a constant ratio to the latus rectum.

Corollary.

If a sphere be described about the vertex of the cone as centre, the latus rectum of the section made by any plane touching it will be constant, and will be equal to the diameter of the circular sections whose planes touch the sphere.

PROPOSITION VIII.

79. *The sphere of which the line joining the centres of the focal spheres of any section is a diameter contains the auxiliary circle of the section.*

(i) Since fF (in Art. 78) subtends right angles at A and A', the sphere on fF as diameter cuts the plane of section in the circle on AA' as diameter, which is the auxiliary circle of the section.

The annexed duplicate proof further establishes the relation between the auxiliary circle and the tangent.

(ii) Through any tangent YZ to the section draw a plane through f and likewise a plane through F. These bisect the supplementary angles between the plane of section and the tangent plane through YZ to the cone, and are therefore *at right angles.*

If SY and HZ be the focal perpendiculars upon the tangent, fY is at right angles to YZ and to the plane FYZ.

Hence fY is at right angles to FY, and the sphere on fF as diameter passes through Y, and its trace on the plane of section is a circle, whereof AA' is evidently a diameter.

Corollary.

The right angled triangles fSY and FHZ being similar,

$$SY.HZ=fS.FH=CB^2,$$

or the product of the focal perpendiculars upon the tangent is constant.

SCHOLIUM B.

The constructions for the Foci and Directrices of the sections of the cone are due to Hamilton, Dandelin, and others.

HUGH HAMILTON of Dublin, in Lib. II. prop. 37 of his work entitled *De Sectionibus Conicis Tractatus Geometricus in quo ex natura ipsius Coni Sectionum Affectiones facillime deducuntur methodo nova* (Londini, 1758), establishes the following properties. In the figures of Art. 75 (supposing the spheres to be omitted) if S be a focus and AE be taken equal to AS, then (1) the S-directrix is determined by the intersection of the plane of the conic with the plane of circular section EQE'; and (2) the vector SP to any point P of the conic is equal to the segment PQ cut off by the same circular section from the side OP of the cone; and (3) when the conic is bifocal two circular sections are thus determined which intercept on any side of the cone a length QR equal to the transverse axis. Having thus established the equality of AS, AE and of $A'S$, $A'E'$, as well as the equality of OE and OE', he had virtually proved that the focus S might be determined as the point of contact of AA' with the inscribed circle or one of the escribed circles of the triangle OAA', or in other words as the point of contact of a Focal Sphere with the plane of section. He did not however state his conclusion in this form, but presupposed the determination of S by the relation $AS.A'S=CB^2$, and then proved MX to be the directrix by shewing that $CS:CA=SA:AX$.

QUETELET contributed a *Mémoire sur une nouvelle Théorie des Sections Coniques considérées dans le Solide* (presented Dec. 23, 1820) to the *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-lettres de Bruxelles* (tome II. pp. 123—153, 1822), in which he shewed *inter alia* (1) that the foci of a section are determined by the relation $OA \pm OA' = SH$; and (2) that in an elliptic section $OP-SP$ is constant and equal to $OB-CA$. These results, so far as they go, are identical with Hamilton's; but Quetelet (unlike Hamilton) gives no construction for the directrices. In the course of the above *Mémoire* he refers to his tract on the *Curva Focalis*, or "Focale" (Gandavi, 1819).

DANDELIN, in a *Mémoire sur quelques propriétés remarquables de la Focale Parabolique* published in the same volume of the *Nouveaux Mémoires* (II. 171—202), begins by inscribing the FOCAL SPHERES and thus determining the foci of the sections. In *tome* III. (1826) he extends the same construction to the Hyperboloid of revolution, but in neither case does he make any mention of the directrices. See also *tome* IV. 77; Quetelet's *Correspondance mathématique et physique*, I. 82; Gergonne's *Annales*, XV. 392.

A complete determination of the Foci and Directrices of the sections of the cone by means of the Focal Spheres was at length proposed by PIERCE MORTON (B.A., 1825) before the *Cambridge Philosophical Society* in 1829 (*Transactions*, vol. III. pp. 185—190, 1830; and see pp. 228—9 of the anonymous *Geometry, Plane Solid and Spherical*, in the Library of Useful Knowledge, London, 1830). From his introductory remarks it would seem that he was not acquainted with the investigations of Hamilton and Dandelin.

THE SEGMENTS OF CHORDS.

PROPOSITION IX.

80. *A chord of a cone being divided at any point, to determine the rectangle contained by its segments.*

Upon the surface of a right or scalene cone take *any* two points A and A' (figures of Art. 75), and in the line AA' or its complement take any point X: it is required to determine the magnitude of the rectangle $XA \,.\, XA'$.

Take any fixed circular section KRK', and from the vertex O of the cone draw a parallel to AA' to meet its plane in Z; and let the plane $AOZA'$ meet the fixed circular section in K and K' and the parallel section made by a plane through X in E and E'.

Then by similar triangles AEX, ZKO and $A'E'X$, $ZK'O$,

$$XA \,.\, XA' : XE \,.\, XE' = ZO^2 : ZK \,.\, ZK',$$

where (1) the latter ratio depends only upon *the direction of* AA', since when this is given the point Z is known and the product $ZK \,.\, ZK'$ is determined, and (2) the magnitude of $XE \,.\, XE'$ depends only upon *the position of* X in space.

By taking A and A' on a given plane of section we deduce the results of Art. 16.

SCHOLIUM C.

The method of Prop. IX. is frequently attributed to Hamilton, in whose treatise it holds a prominent place (*Sectiones Conicæ*, 1758, Lib. I. props. X. XI.). He supposed that he had thereby settled the old controversy about the cone and the plane in favour of the ancients, who derived the "sections" from the cone; not being aware that the property in question could be proved *in plano* with at least equal ease (Art. 16). Hamilton's proposition had appeared in the preceding century in the *Tractatus* XXIV. *De Sectionibus Conicis*, props. 48, 49 (pp. 419—20) appended to EUCLIDES ADAUCTUS *&c. auctore D. Guarino Guarino* (Augustæ Taurinorum, 1671); where in prop. 48 the case of two parallel chords cut by a single chord is considered, and the property of two pairs of parallel chords is deduced in prop. 49. See also *Synopsis Palmariorum Matheseos: or A New Introduction to the Mathematics*, by W. Jones (London, 1706), Part II. § 69 (3), p. 255: "In any Conic Section, if two Parallels are cut by two others, and all terminate at the Curve, the Rectangles of the Segments shall be Proportional." These references are given by Abr. Robertson, *Sectionum Conicarum libri septem*, p. 348 (Oxon. 1792).

EXAMPLES.

541. The latus rectum of a parabola cut from a given right cone varies as the distance between the vertices of the section and of the cone.

542. The foci of all similar sections of a given right cone lie upon two other right cones.

543. Prove by means of Art. 75, Cor. that the parallel sections of a cone have the same eccentricity; and give a construction for cutting an ellipse of given eccentricity from a given right cone.

544. The eccentricity of a section is a ratio of majority or minority according as the acute angle between the axes of the cone and of the section is less or greater than half the vertical angle of the cone.

545. If two or more plane sections have the same directrix, the corresponding foci lie on a straight line through the vertex of the cone.

546. Shew how to cut from a given right cone an ellipse whose axes shall be of given lengths, or whose latus rectum and area shall be of given magnitude.

547. The area of *the triangle through the axis**, in the right

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* APOLLONIUS supposed his sections to be made by planes at right angles to the plane drawn through the axis of the cone (defined as on p. 192) at right angles to its base. The *triangle through the axis* (viz. of the cone) was the triangle determined by the vertex of the cone and the trace of the conic upon the plane through the axis. See Chasles, *Aperçu Historique*, p. 18 (ed. 2, Paris, 1875).

cone, varies as the square of the minor axis of the section, and the volume cut off from the cone by the plane of the section varies as the cube of its minor axis, and is of constant magnitude so long as the minor axis is constant. Is the same true of the scalene cone?

548. If the minor axis of a section be given in length and direction, the locus of the centre of the section is a hyperbola.

549. Given a right cone and a point within it, construct the two sections which have the point for a focus; and shew that their planes make equal angles with the straight line joining the point to the vertex of the cone.

550. The vertical angle of a right cone being a right angle, the perpendicular from the vertex to any plane is equal to the semi-latus rectum of the section made by that plane.

551. Shew how to cut a section of maximum eccentricity from a given cone. Under what conditions is it possible to cut a rectangular hyperbola from a cone?

552. Shew how to place a given section (when possible) in a given cone.

553. If from the centre C of a hyperbolic or elliptic section a line CVV' be drawn at right angles to the axis to meet the sides of the triangle through the axis, the square of the semi-axis conjugate is equal to $CV.CV'$; and the semi-axis conjugate is equal to the distance of C from the vertex of the cone in the case in which the transverse axis of the section is parallel to the axis of the cone.

554. Two right cones of supplementary vertical angles being placed with their axes at right angles and their vertices coincident, shew how to cut from them a pair of conjugate* hyperbolas. Shew also that if the two cones be cut by a plane perpendicular to their common generating line, the directrices of one of the sections will pass the foci of the other.

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* That is to say, conjugate in form and dimensions, but *not* lying in the same plane.

555. The locus of the centres of the elliptic sections whose major axes are equal is a prolate or oblate spheroid.

556. If V be the vertex of a right angled cone, and PN the ordinate of P in a parabolic section whose vertex is A, shew that the semi-latus rectum is equal to $VP-AN$.

557. An ellipse and a hyperbola lying in planes at right angles are so situated that the foci of each are at the vertices of the other. Shew that if S be the vertex and A the focus and P any point upon a branch of the hyperbola, and if Q be any point on the ellipse, then

$$AS+PQ=AP+SQ.$$

558. If two cones be described touching the same two spheres, the eccentricities of their sections by identical planes are in a constant ratio.

559. The vertex of a right circular cone which contains a given ellipse lies on a certain hyperbola, and its axis touches the hyperbola, and conversely.

560. Two parallel tangents to a section of a right cone meet in M and M' a plane which touches the focal spheres in points Q and Q' on a generating line: shew that a circle goes round $MQM'Q'$.

561. Determine the asymptotes of a given section of a scalene cone.

562. Assuming that one focus of the shadow of a sphere standing on a horizontal plane and exposed to the light of the sun is its point of contact with the plane; find the envelope of the corresponding directrix, and the locus of the remaining focus, for a given day and place.

563. By properties of the cone, or otherwise, find the locus of the extremity of the shadow of a vertical gnomon erected on a horizontal plane, on a given day and in a given latitude.

564. The centre of a sphere moves in a room a vertical plane which is equidistant from two candles of the same height from the floor: determine its locus if the shadows upon the ceiling be always in contact.

565. If a point move in a plane in such a way that the sum or difference of its distances from two fixed points, one of which lies in the plane and the other without it, is constant, its locus will be a plane section of a right circular cone whose vertex is at the external given point.

566. Prove in Art. 75 that SE and SE' bisect the diameters of the circular sections through A' and A respectively; and that SO, AE', $A'E$ cointersect, and hence that S and $E'E$ produced divide AA' harmonically.

567. Prove from the cone that the intercept on any tangent to a conic between the curve and a directrix subtends a right angle at the corresponding focus.

568. Prove also that the tangents from any point to a section subtend equal or supplementary angles at either focus.

569. A tangent to a right cone being drawn, there may always be drawn through it pairs of planes cutting the cone in sections which have equal parameters.

570. The section of maximum parameter which can be drawn through a given point on a right cone has its plane at right angles to the generating line through the point, and has its tangent at that point parallel to the base of the cone.

571. Through a given point on a right cone there may be drawn any number of planes making sections which have equal parameters; and the envelope of these planes is another right cone, having its vertex at the given point and its axis coincident with the side of the original cone through that point.

572. In a given right cone, the locus of the foci of all equal parabolas is a circle whose plane is parallel to the base; the locus of the foci of all the parabolas whose planes are parallel is a straight line through the vertex of the cone; and the locus of the foci of all the parabolas that can be drawn in the cone is another right cone having the same axis.

573. The sphere inscribed in a right cone so as to pass through a vertex of a section intercepts upon the axis of the section a length equal to its latus rectum.

574. If sections of a right cone be made having one of their vertices at a fixed point on the cone, their circles of curvature at that vertex lie upon a sphere inscribed in the cone.

575. The sum or difference of the tangents from any point on a conic to the circles of curvature at its vertices is constant.

576. Prove from the right cone that a conic section may be regarded as the locus of a point such that the sum or difference of the tangents therefrom to two fixed circles is constant.

577. Prove from the right cone that a conic section is the locus of a point such that the tangent therefrom to a fixed circle is in a constant ratio to its perpendicular distance from a fixed straight line; and prove that in the case in which the straight line cuts the circle it is the chord of real *double contact* of the circle with the conic;* and prove that the above-mentioned constant ratio or *modulus* is equal to the eccentricity of the conic.

578. Two circles have double internal contact with an ellipse,† and a third circle passes through the four points of contact. If t, t', T be the tangents from any point on the ellipse to these three circles, prove that $T^2 = tt'$.

579. Notice the forms assumed by the several properties of the acute-angled cone when its vertical angle is diminished indefinitely, so that the surface becomes a cylinder.

580. An oblique cone or cylinder being described upon a circular base, shew that its subcontrary sections are likewise circular.‡

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* Hence it appears that *a focus of a conic may be regarded as an evanescent circle having double contact with the conic* at the two imaginary points in which it is intersected by the corresponding directrix; and it may be inferred that the lines joining the focus to the two imaginary points at infinity through which all circles pass are tangents to the conic, and hence that *all conics which have the same two foci may be regarded as inscribed in the quadrilateral which has the two foci and the two circular points at infinity for its opposite vertices.* See SALMON'S *Conic Sections*, chap. XIV, on *Methods of Abridged Notation.*

† If CM be the central abscissa of a point of contact, and CN the abscissa of any point on the conic, the tangent from that point to the circle is equal to $\frac{CM}{e} - e.CN$.

‡ Two sections are said to be *subcontrary* when the traces of their planes upon the plane of reference are inclined to the sides of the cone or cylinder at angles which are alternately equal.

581. Extend the theorems of Art. 72 to the oblique cone; and shew that if through any point on a side of the triangle through the axis there be drawn two planes of circular section and any other plane *between* them, the third plane will cut the cone in an ellipse, having its *minor axis* in the plane of reference.

582. If AA' be the axis of a section of a scalene cone, and AD and Ad be diameters of its circular sections through A, shew that the square of the distance between the foci is equal to $A'D \, . \, A'd$. Shew also in the right cone, with the construction of Art. 72, that the circle drawn with the middle point of AA' as centre to bisect AH and $A'K$ passes through the foci.

583. If a scalene cone be cut by any plane at a given distance from its vertex, the latus rectum of the section will be constant.*

584. The *transverse axis* of a section of an oblique cone being supposed to lie in the plane of reference, prove that the circle which touches the axis and passes through the centres of the two circular sections which can be drawn through either vertex determines the nearer focus.†

585. If the *conjugate axis* lie in the plane of reference, and if two circular sections be drawn through either of its

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* This extension of Art. 78, Cor. may be proved as follows. Supposing the transverse axis to lie in the plane of reference, let DD' be the diameter (in that plane) of the section which is parallel to the base and equidistant with the plane of the conic from the vertex O (fig. Art. 72) of the cone. Draw OYZ parallel to AA' to meet AH in Y and DD' in Z. Then it may be shewn that $AY = OZ$, and hence that $DD' : AH = A'K : AA'$. The latus rectum is therefore equal to DD'. This theorem is due to James Bernouilli. See the Leipzig *Acta Eruditorum*, ann. 1689, pp. 586–8; and Chasles' *Aperçu Historique*, p. 19.

† Exx. 584 and 585 are from Chasles' *Aperçu Historique*, Note IV, p. 285, where it is added that the *Eccentricity* is a mean proportional (Ex. 582) to the distances of the centre of the conic from the centres of the two circular sections through either of its vertices. It is to be noted that (before the directrix came into general use) the eccentricity was sometimes defined (1) as the *distance* of the foci from the centre, or (2) as the ratio of that distance to the semi-axis, or (3) as the ratio $\frac{b-a}{a}$, where b denotes the semi-latus rectum and a the perihelion distance of the orbit (Euler's *Theoria Motuum Planetarum et Cometarum*, prob. VIII. cor. 1, p. 86).

extremities, the circle upon the line joining their centres as diameter passes through the foci.

586. Every tangent plane to a cone cuts the cyclic planes in a pair of lines making equal angles with the line of contact; and every plane through two sides of the cone cuts the cyclic planes in two lines which make equal angles with those sides.

587. The sum or difference of the angles which any tangent plane makes with the cyclic planes is constant.

588. The sum or difference of the lines drawn from the vertex of a right cone to the extremities of any diameter of a section is double of the line from the vertex to an extremity of the conjugate axis.

589. If a sphere drawn through the vertex of an oblique cone cuts the cone in a circle, the plane of a section subcontrary thereto cuts the planes of any two great circles of the sphere in a pair of lines inclined at the same angles as their planes.

590. The lines of intersection of any tangent plane to a scalene cone with its two cyclic planes are such that the product of the tangents of the angles which they make with the intersection of the cyclic planes is constant.

591. The product of the sines of the angles which any side of a cone makes with the two cyclic planes is constant.

592. If a section of a cone be made by a plane which cuts the planes of two subcontrary circular sections and the sphere containing them in two right lines and a circle respectively, and if a chord be drawn to the circle from any point of the section; the product of the segments of the chord is to the product of the perpendiculars from the assumed point to the subcontrary planes in a constant ratio.*

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* For this very general theorem and its corollaries see the article by Mr. JOHN WALKER on *Geometrical Propositions relating to Focal Properties of Surfaces and Curves of the Second Order* in the *Cambridge and Dublin Mathematical Journal*, vol. VII. pp. 16—28 (1852). The special case of Ex. 594 is also proved in the *Messenger of Mathematics*, vol. IX. pp. 33, 34. See also Mr. S. A. Renshaw's treatise on *The Cone and its Sections* (London, 1875).

593. The above-mentioned constant ratio is the ratio of the product of the sines of the angles which the cyclic planes make with the plane of the section to the product of the sines of the angles which they make with any side of the cone.

594. A sphere being drawn through two subcontrary circular sections of a cone, and the planes of those sections being produced to meet; prove that their line of intersection is a directrix of the section made by either plane drawn through it to touch the sphere, and that the point of contact is the corresponding focus.

595. From the above construction deduce that any tangent to a conic (from the curve to either directrix) subtends a right angle at the corresponding focus.

596. All the right cones which have the same conic section for their base have their vertices upon another conic section, lying in a plane at right angles to that of the former, the foci of each curve being at the vertices of the other.

597. If a hyperboloid of revolution and its asymptotic cone be cut by a plane, their two sections will be similar.*

598. If two spheres be inscribed in a conoid† so as to touch a given plane of section, the two points of contact will be the foci of the section.

599. All sections of a conoid made by planes through one its foci have that point for one of their foci, and they have the intersections of those planes with the directrix plane of the conoid for their corresponding directrices.

600. The cone whose vertex is at a focus and whose base is any plane section of a conoid is a right cone.

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* For proofs of the theorems of Exx. 597—600 see Hutton's *A Course of Mathematics*, composed for the use of the Royal Military Academy, vol. II. pp. 196—203 (12th edition, ed. Thomas Stephens Davies, 1843); and see Besant's *Conic Sections treated geometrically*, chap. XII.

† A *conoid* is the surface generated by the revolution of a conic about one of its axes.

CHAPTER VIII.

CURVATURE.

81. If PQ be a small arc of a curve and PT its tangent at P, the angle QPT is called the *Angle of Contact* of the arc PQ.

If PQ' be an arc of a second curve touching the former at P, and if QQ' meet the common tangent in T; then will the curvature of PQ at P be equal to or greater or less than the curvature of PQ' at P according as the limiting ratio of the angle of contact QPT to the angle of contact $Q'PT$ (when PQ and PQ' are diminished indefinitely) is a ratio of equality, majority, or minority.

The *Circle of Curvature* of a conic at any point P is the circle which has the same curvature as the conic at P: it is therefore the limiting position of the circle which touches the curve at P and meets it again at an adjacent point which ultimately coalesces with P: it is also the limiting position of the circle which meets the curve at P and at two other points which ultimately coalesce with P. The centre, radius and diameter of this circle are called the *Centre of Curvature*, the *Radius of Curvature*, and the *Diameter of Curvature* of the conic at P, and its chord in any direction through P is called the *Chord of Curvature* of the conic in that direction.

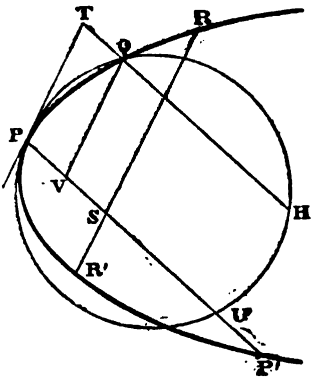
It is easily seen that a circle which cuts a conic must cut it in two or four points, and hence that the circle of curvature at any point P of a conic will in general *cut* as well as touch it at P, and will also cut it in one other point. Any other circle touching the conic at P must lie wholly within or without the former, and since it cannot cut the conic at P it is easily seen that it cannot pass *between* the curve and its circle of curvature at that point. The circle of curvature is therefore the circle of closest contact with the conic at P, and is called its *Osculating Circle* at that point.

A circle which touches a conic at an extremity of either axis will in general meet it again at the two extremities of a chord parallel to the other axis; whence it readily follows that the circle of curvature at an extremity of an axis is to be regarded as meeting the conic in *four* coincident points. It may also be regarded as having double contact with the conic at two coincident points,* and it does *not* cut the curve at its point of contact. It is easy to determine the points in which any other circle touching the conic at the same point meets it again, and hence to shew that no such circle can pass between the conic and its circle of curvature at that point.

PROPOSITION I.

82. *The focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at that point.*

Let PSP' be any focal chord of a conic, PT the tangent at P, and RSR' the focal chord parallel to PT.



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* By regarding the centre of curvature at the vertex as the ultimate position of the foot of the normal at P when P coalesces with A, we deduce from the property $SG = e\,.\,SP$ (Art. 10) that the radius of curvature at A is equal to $AS\,(1+e)$, or to the semi-latus rectum. By like considerations it may be shewn that the radius of curvature at B in the ellipse is equal to $\dfrac{CA^2}{CB}$.

Describe a circle touching the conic at P and cutting it at an adjacent point Q, and let TQ be taken parallel to PP', and let it be produced to meet the circle again in H and the conic in Q'. Then by Art. 16 and by a property of the circle,

$$TQ.TH : TQ.TQ' = TP^2 : TQ.TQ' = RR' : PP',$$

or

$$TH : TQ' = RR' : PP'.$$

Let PS meet the circle again in U; and let the point Q coalesce with P, so that the circle becomes the circle of curvature at P.

Then T likewise coalesces with P, and TQ' with PP', and TH with PU; and therefore (from the above proportion) the chord of curvature PU (being the limit of TH) is equal to the focal chord RR' parallel to PT, as was to be proved.*

Corollary 1.

It follows that in a central conic the chord of curvature at P through *either* focus is equal to $\frac{2CD^2}{CA}$ (Art. 36, Cor.): in the parabola the chord of curvature through the focus, or parallel to the axis,† is equal to $4SP$: and in the general conic the focal chord of curvature is equal to $\frac{2PG^2}{L}$ (Art. 15, Cor. and Ex. 45).

Corollary 2.

Given the chord of curvature at P in any one direction, the chord in any other direction can be determined. For let PU be the given chord and PV any other, and let a parallel to the tangent at P meet PU in K and PV in F; then it is evident from the circle that

$$PV : PU = PK : PF,$$

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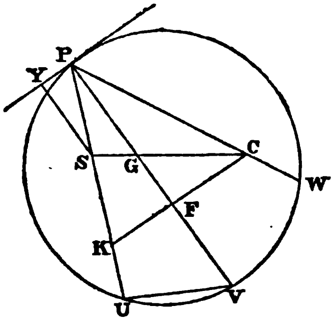
* This proof is due to Professor TOWNSEND (Salmon's *Conic Sections*, Art. 397): a variation of it will be given in Prop. II., where the circle of curvature is regarded as the limit of a circle which cuts the conic in three points, two of which ultimately coalesce with the third.

† Draw a circle touching the parabola in P and cutting it in R (fig. Art. 32), and let MR meet the circle again in K. Then $MR.MK = MP^2 = 4SP.MR$ (Art. 28, Cor. 2), or $MK = 4SP$. Hence another proof of the result given above in the text. It is to be noted that any two chords of the circle of curvature equally inclined to the normal at P are equal.



or the ratio of PV to the given chord PU is known. If K be supposed to coincide with S, then

$$PV.PF = SP.PU = SP.RR',$$

which gives an expression for the chord of curvature PV in any direction in terms of the focal chord RR' (of the conic) parallel to the tangent at P. For example, in the parabola it follows at once that the *Diameter of Curvature* is equal to $\frac{4SP^2}{SY}$, where SY is the focal perpendicular upon the tangent. In the central conics, supposing KF to pass through C, we deduce that the *Diameter of Curvature* is equal to $\frac{PU.CA}{PF}$ (PU being drawn through S), or $\frac{2CD^2}{PF}$, or $\frac{2CD^3}{CA.CB}$, or $\frac{2PG^3}{L^2}$.

PROPOSITION II.

83. *At any point of a conic the chord of curvature in any direction is to the chord of the conic in the same direction as the focal chords (of the conic) parallel to the tangent and to the chord of curvature.*

Let a circle meet a conic in three adjacent points Q, P, Q',[*] and let PU be a chord of the circle, and let it meet QQ' in V and the conic again in P'.

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[*] Complete the chord QVQ' in the figure of Prop. I.

Then by a property of the circle and by Art. 16,

$$PV.VU : PV.VP' = QV.VQ' : PV.VP' = q : p,$$

where p and q are the focal chords of the conic parallel to PP' and QQ'.

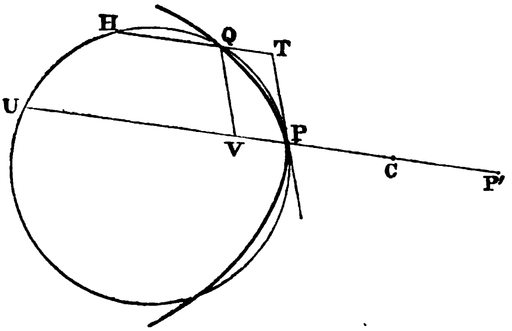
Therefore $$VU : VP' = q : p.$$

Hence, if Q and Q' (and therefore also V) be supposed to coalesce with P, so that the circle becomes the circle of curvature and QQ' becomes the tangent at P, it follows that the chord of curvature PU is to the chord of the conic PP' in the same direction as its focal chord parallel to the tangent at P to its focal chord parallel to PP', as was to be proved.*

PROPOSITION III.

84. *To determine the length of the central chord of curvature at any point of an ellipse or hyperbola, and likewise the length of the chord of curvature drawn in any other direction.*

(i) Let PCP' be any diameter of a central conic, QV a double ordinate of that diameter adjacent to its extremity P: we have to evaluate the central chord of curvature of the conic at P.



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* If any focal chord meet the tangent at P in T, it follows at once by Ex. 79 (cf. Ex. 447, note) that the chord of curvature at P parallel to it is equal to $\frac{SP.RR'}{ST}$, as was virtually shewn in Art. 82, Cor. 2. It will be observed that Prop. II. alone completely determines the curvature of a conic at any point, but it seemed desirable to regard the subject from different points of view.

Draw a circle touching the conic at P and cutting it at Q, and let it meet PP' again in U, and let its chord HQ parallel to PP' meet the tangent at P in T.

Then since $$QV^2 = TP^2 = TQ\,.\,TH,$$

therefore $$TQ\,.\,TH : PV\,.\,VP' = CD^2 : CP^2,$$

or $$TH : VP' = CD^2 : CP^2,$$

where CD is the semi-diameter conjugate to CP.

By making Q coalesce with P, so that the circle becomes the circle of curvature at P, we deduce that

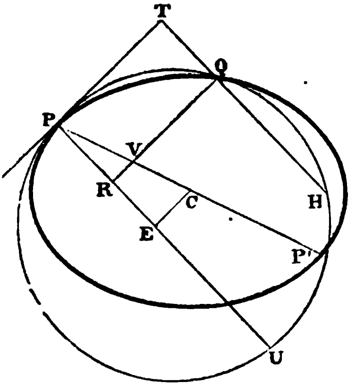
$$PU : PP' = CD^2 : CP^2,$$

or $$PU\,.\,CP = 2\,CD^2,$$

where PU is the central chord of curvature at P.

(ii) More generally, if QH and PU be parallel chords of the circle drawn in any direction, and if QV meet PU in R and CD in E, then since $QR : QV$ is a ratio of equality when Q coalesces with P,[*] it is easily seen, as in the former case, that ultimately

$$PR\,.\,TH : PV\,.\,VP' = CD^2 : CP^2;$$



and it follows by parallels that

$$PE\,.\,TH : CP\,.\,VP' = CD^2 : CP^2,$$

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[*] See NEWTON'S *Principia*, Lib. I. Sect. I. lemma 7, cor. 2. Also, upon curvature in general, see lemma 11 with the notes in the edition by Mr. P. T. Main (after J. H. Evans) of Sections I–III, IX, XI; and see the *Appendix*, pp. 181–141, in that edition (Cambridge, 1871).

or (TH being now coincident with the chord of curvature PU)

$$PU \, . \, PE = 2\,CD^2.$$

The *Diameter of Curvature* at P is equal to $\frac{2\,CD^2}{PF}$, where PF (defined as in Art. 46) is equal to the central perpendicular upon the tangent.*

PROPOSITION IV.

85. *To evaluate the common chord of a conic and its circle of curvature at any point.*

If three of the four points in which a circle meets a conic coalesce at P and their fourth common point be Q, it follows from Art. 16, Cor. 2 that the chord PQ and the tangent at P are equally inclined to the axis; that is to say, *the common chord of a conic and its circle of curvature at any point and their common tangent at that point are equally inclined to the axis.*

(i) In the Parabola, let the common chord PQ meet the axis in R, and let the common tangent at P meet the axis in T. Then it is easily seen that the second tangent TP' from T is parallel to PQ, and that PP' is a double ordinate to the axis, and that PQ is equal to four times PR or TP.†

(ii) In the Ellipse, let the diameter CD parallel to the tangent at P meet the common chord PQ in X; then will $PX \, . \, PQ$ be equal to $2\,CD^2$.

Take a circle of radius CA, and take any diameter of the same, and draw the chord PQ making the same angle therewith as the tangent at P, and let the diameter parallel to that tangent meet the chord in X.

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* The radius of curvature of the general conic may be evaluated by regarding the centre of curvature as the ultimate intersection of two consecutive normals, and assuming that $SG = e \, . \, SP$ (Art. 10), and $PK = \frac{1}{2}$ latus rectum (Art. 11). The expression $\frac{PG^3}{L^2}$ (Art. 82, Cor. 2) for the radius of curvature has been obtained in this direct manner by Professor ADAMS (*Oxf. Camb. Dubl. Messenger of Mathematics*, vol. III. pp. 97–99).

† The chord of curvature in any other direction may be deduced. For example, it may be shewn by angle-properties that the circle meets PS produced in a point V lying on the diameter through Q in the parabola, and hence that PV is equal to $4SP$.

Then if PC meet the circle again in P',

$$PX \,.\, PQ = PC \,.\, PP' = 2CA^2.$$

It follows by Orthogonal Projection that in the Ellipse (the same letters being used) $PX \,.\, PQ$ is equal to twice the square of the semi-diameter parallel to PQ: therefore, the diameters parallel to PQ and the tangent at P being equal,

$$PX \,.\, PQ = 2CD^2.$$

It is left to the reader to obtain the same result by a method applicable to the hyperbola also.*

SCHOLIUM.

APOLLONIUS in treating of maxima and minima takes a point on the axis of a conic at a distance equal to the semi-latus rectum from its vertex (*Conics*, Lib. V. props. 4—6, &c.), which is in fact the centre of curvature at the vertex, although he does not in any direct manner touch upon the subject of curvature. Cf. Vincentio Viviani's treatise, *De Maximis et Minimis geometrica divinatio in quintum Conicorum Apollonii Pergæi adhuc desideratum* (Florent. 1659); and see Abr. Robertson's *Sectionum Conicarum libri septem*, p. 372 (Oxon. 1792).

HUYGHENS came very near to the subject of curvature when he propounded his theory "De linearum curvarum evolutione et dimensione" (*Horologium Oscillatorium*, Pars III. Paris. 1673), since the evolute of a curve is the envelope of its normals and the locus of its centres of curvature. The first case of rectification of a curve was that of the cubical parabola by William NEIL:† the cycloid was rectified soon after by Christopher WREN. Huyghens had previously (1657) reduced the rectification of the parabola [Ex. 535, note] to the quadrature of the hyperbola (*Horolog. Oscill.* pp. 72, 77).

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* The proof in the text is given to indicate the applicability of Orthogonal Projection (chap. IX.) to the treatment of curvature. If any two lines (as PQ and the tangent at P) equally inclined to the axis of the ellipse be projected on to its auxiliary circle, the projected lines will be equally inclined to the axis; or if the term *equally* inclined be restricted to parallels, we must say that if two lines make equal or supplementary angles with the axis, their projections will make equal or supplementary angles with the axis.

† This was suggested by the *Arithmetica Infinitorum* of WALLIS (Oxon. 1656): Van-Heuraet seems to have rectified the same curve independently (1659) very soon after Neil. See Montucla's *Histoire des Mathématiques*, Part IV. Liv. VI. § II, tome II. p. 808. Huyghens awards the palm to the later discoverer, considering that Neil's investigation was incomplete (*Horolog.* p. 72).

The *Osculating Circle* received its name from LEIBNITZ: see *Meditatio nova de natura Anguli Contactus*[*] *& Osculi* by G. G. L. in the *Acta Eruditorum* for 1686 (pp. 289—292, by misprint 489—292). See also pp. 292—300, and the *Acta* for 1692, pp. 30—35, 110—116, 440—6. The subject of curvature was considered by NEWTON in the seventh and eighth chapters of his *Geometria Analytica.*

EXAMPLES.

601. Prove the following construction for the centre of curvature at any point P of a conic. From the point in which the normal meets the axis draw a perpendicular to the normal meeting SP in Q, and from Q draw a perpendicular to SP meeting the normal in O. Then O is the centre of curvature at P.

602. The circle which touches a conic at P and intercepts upon the diameter through P a length equal to its parameter[†] is the circle of curvature at P.

603. The circle of curvature at any point of a conic being the circle through that point and two others which ultimately coalesce with it, shew that the centre of curvature may be regarded as the point of ultimate intersection of two consecutive normals to the conic.

604. The circle of curvature at any point of a conic may be regarded as touching three consecutive tangents to the conic[‡]

605. Determine the position of the common chord of a parabola and its circle of curvature at an extremity of the latus rectum.

606. The diameter at either extremity of the latus rectum of a parabola passes through the centre of curvature at its other extremity.

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[*] For earlier controversies on the nature of the angle of contact see WALLIS, *De Angulo Contactus et Semicirculi disquisitio geometrica* (Oxon. 1656).

[†] The parameter of any diameter of a central conic is defined as a third proportional to that diameter and its conjugate.

[‡] For proofs of Exx. 603—4 see the section on curvature in Main's NEWTON, *Appendix* (see above p. 219, note).

607. The radius of curvature at any point of a parabola is double of the portion of the normal intercepted between the curve and the directrix.

608. At any point of a parabola the intercept made by the circle of curvature upon the axis is a third proportional to the latus rectum and the parameter of the diameter to the point.

609. At any point P of a parabola, if PY be the projection of SP upon the tangent, the chord of curvature through the vertex is a third proportional to AP and $2PY$.

610. If R be the middle point of the radius of curvature at P in a parabola, PR subtends a right angle at S.

611. If the normal to a parabola at P be produced to any point O, express the radius of curvature at P in terms of the line OP and its inclinations to the tangents from O.

612. The radius of curvature at an extremity of the latus rectum of a parabola is equal to the projection upon the directrix of the focal chord parallel to the tangent at that point.

613. Given a circle and a straight line, it is required to find a parabola (see Ex. 607) having the line for its directrix and the circle for a circle of curvature.

614. The envelope of the common chords of a parabola and its several circles of curvature is a parabola, and the locus of their middle points is a parabola.

615. The tangent from any point of a parabola to the circle of curvature at its vertex is equal to the abscissa of the point.

616. If tangents TQ and TQ' be drawn to a parabola from any point T on the fixed diameter which meets the curve in P and the directrix in M, the centre of the circle round TQQ' lies at a constant distance $2PM$ from the directrix. If O be the centre of the circle, D the projection of O upon the fixed diameter V the middle point of QQ', and X the foot of the directrix, shew that

$$OD : TM = OD : DV = MX : SX.$$

Shew also that the radius of the circle varies as ST.

617. If from the vertex of a parabola chords AR and AR' be drawn equally inclined to the axis, the normals at the extremities of any chord parallel to AR intersect upon the normal at R';[*] and the centre of the curvature at the extremity of the diameter which bisects AR lies upon the normal at R'.

618. Triads of points can be found on a parabola such that the normals thereat cointersect. The circle through any three such points passes through the vertex of the parabola, and the centroid of the triangle which they determine lies on the axis.

619. In Ex. 616 shew that the ordinate of the point of concourse of the normals at Q and Q' is to the ordinate of V as the product of the ordinates of Q and Q' to the square of the semi-latus rectum. Hence determine the ordinate of the centre of curvature at P, and likewise the length of the radius of curvature.

620. Give a geometrical method of drawing normals to a parabola from a point on the curve.

621. Determine a point on a given conic at which the circle of curvature is of given magnitude; and in the case of the ellipse determine the limits of its magnitude.

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[*] Let QQ' in Ex. 616 be supposed parallel to AR. Produce TO to meet the circle again in N (the point of concourse of the normals at Q, Q'), draw the perpendicular NH to the fixed diameter, and take HK equal to TM upon MH produced. Then it is easily seen that K is *a fixed point* (the position assumed by N when QQ' passes through S), and $NH : HK = 2MX : SX$. Therefore N lies on a fixed straight line through K, which may be proved (from the property of the subnormal) to be the normal at R'. For the above I am indebted to the Rev. A. F. Torry, Fellow of St. John's College. The following method may also be suggested: since the angle OST is a right angle (Ex. 146), we have an intercept TO between two fixed straight lines DO and DT at right angles subtending a right angle at a fixed point S, and we have to shew that the extremity N of TO produced to double its length lies upon a fixed straight line. Or again, taking three positions of P in Art. 26—P and P' on one side and P'' on the other side of the axis, and supposing the normals at the three points to cointersect at a point whose projection upon ths axis is Z, we easily prove that $PN . ZG = P'N' . ZG' = P''N'' . ZG''$; whence it follows, after some reductions, that $PN + P'N' = P''N''$. On the tetrads of concurrent normals to a central conic see Ex. 286.

622. If the osculating circle at a vertex of an ellipse passes through the further focus, determine the eccentricity.

623. The circle of curvature at a point on an ellipse passes through a vertex: find the point.

624. The circle of curvature at an extremity of one the equal conjugate diameters of an ellipse passes through its other extremity. Explain the corresponding result in the hyperbola.

625. The circles of curvature at the extremities of two conjugate radii CP and CD of an ellipse meet the curve again in Q and R: shew that PR is parallel to DQ.

626. Find the points on a central conic at which the diameter of curvature is a mean proportional to the axes.

627. Express the chord of curvature perpendicular to the axis at any point of an ellipse in terms of the ordinate of the point.

628. From the point in which the tangent to an ellipse at P meets the axis a straight line is drawn bisecting one of the focal distances and meeting the other in Q. Prove that PQ is one fourth of the focal chord of curvature at P.

629. If the circle of curvature at P in an ellipse passes through a focus, then P lies midway between the minor axis and the further directrix, and the parallel to the tangent through the focus divides the diameter at P in the ratio of three to one. The circle of curvature cannot pass through either of the foci if the semi-axis exceed the distance between them.

630. Shew that a conic can be described with a given focus so as to have a given circle of curvature at a given point.*

631. The foci of all the ellipses which have a common maximum circle of curvature at a given point lie on a circle.

632. The tangent at P in an ellipse meets the axes in H and K, and CP is produced to meet the circle round CHK in L: prove that $2PL$ is equal to the central chord of curvature at P, and that $CL.CP$ is constant.

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* This is a limiting case of the problem to describe a conic of which a focus and three points are given.

633. A hyperbola which touches an ellipse and has a pair of its conjugate diameters for asymptotes has the same curvature* as the ellipse at their points of contact.

634. At any point P of a rectangular hyperbola, if PN be a perpendicular to an asymptote, the chord of curvature in the direction PN is equal to $\frac{CP^2}{2PN}$.

635. At any point of a rectangular hyperbola the radius of curvature varies as the cube of the radius of the curve.

636. At any point of a rectangular hyperbola the normal chord is equal to the diameter of curvature.

637. At any point of a rectangular hyperbola the diameter of the curve is equal to the central chord of curvature.

638. If from a given point on an ellipse there be drawn a double ordinate to either axis, and if to the diameter through its further extremity a double ordinate be applied from the given point, it will be a chord of the circle of curvature at the given point.

639. The normal chord which divides an ellipse most unequally is a diameter of curvature,† and is inclined at half a right angle to the axis.

640. A chord of constant inclination to the arc of a closed curve divides its area most unequally when it is a chord of curvature.

641. Every chord of a conic which touches the circle of curvature at its vertex is divided harmonically by that circle and the tangent at the vertex.‡

642. The radius of curvature at any point of a parabola is bisected by the circle which touches the parabola at that point and passes through the focus.

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* Curvature is measured by the reciprocal of the radius of curvature.

† The normal in two consecutive positions must cut off equal areas, and must be bisected at the centre of curvature.

‡ Exx. 641–3 are from the *Mathematician*, vol. I. 290 (London, 1845).

643. The radius of curvature at any point of a central conic is cut harmonically by the two circles which touch the conic at that point and pass one through each focus.

644. If T be the point of concourse of the common tangents of an ellipse and its circle of curvature at P, and if O be the centre of the circle, C the centre and S either focus of the ellipse; then T lies on the confocal hyperbola through P, and OC bisects PT, and SP and ST are equally inclined to OS.

645. A circle through the vertex of a parabola cuts the curve in general in three other points, the normals at which cointersect. If the point of cointersection of the three normals describe a coaxal conic having its centre at the vertex of the parabola, the locus of the centre of the circle will be a conic having its centre at the focus of the parabola, and the ratio of its axes will be twice as great as the ratio of the axes of the former conic.

646. Find a point P on an ellipse at which the common tangents of the ellipse and its circle of curvature are parallel. Shew that their common tangents and common chord are parallel to the asymptotes of the confocal hyperbola through P; and that their finite common tangent is bisected by the common chord.

647. The common chord of an ellipse and its circle of curvature at any point and their common tangent thereat divide their further common tangent harmonically.

648. On the normal at P to an ellipse PO is measured outwards equal to the radius of curvature: shew that PO is divided harmonically at the points Q and Q' in which it meets the director circle, and that

$$\frac{1}{PQ} \sim \frac{1}{PQ'} = \frac{2}{OP}.$$

649. The angle between the normals at adjacent points P and Q of an ellipse whose foci are S and S' is equal to $\frac{1}{2}(PSQ + PS'Q)$. Deduce that, if PR be the focal chord of curvature at P,

$$\frac{4}{PR} = \frac{1}{SP} + \frac{1}{S'P}.$$

650. Three points can be found on an ellipse whose osculating circles meet at a given point on the curve:* the three points determine a maximum inscribed triangle, and the four points lie on a circle.

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* Let BCD be a maximum inscribed triangle, and A the point in which the osculating circle at B meets the ellipse again. If B' be a consecutive position of B, the triangles BCD and $B'CD$ must lie between the same parallels. Hence the tangent at B is parallel to CD, which is therefore equally inclined to the axis with AB (Art. 85), so that a circle goes through $ABCD$ (Art. 16, Cor. 2). Ex. 650 (except as regards the area of the triangle) is due to STEINER, who stated the theorem without proof in Crelle's *Journal*, vol. XXXII. 300 (1846): it was proved in vol. XXXVI. 95, by JOACHIMSTAL, who further shewed that the centroid of the three points is at the centre of the ellipse, and that the normals at the three points cointersect.

CHAPTER IX.

ORTHOGONAL PROJECTION.

86. THE foot of the perpendicular from any point in space upon a plane is called the *Orthogonal Projection*, or briefly the *Projection*, of the point upon the plane. The projection of any line or figure upon a plane is the aggregate of the projections of its several points thereupon. The term *Plane Projection* is defined in Art. 90.

The *Parallel Projection* of any figure upon a plane is the aggregate of the points in which a system of parallels from all points of the figure meet the plane: when the parallels are taken at right angles to the plane of projection we come back to the special case of orthogonal projection, to which our attention will be in the first instance confined.

PROPOSITION I.

87. *The projections of parallel straight lines upon any plane are parallel straight lines, and each line or segment is in the same ratio to its projection.*

(i) The projection of a straight line upon a plane is the common section of that plane with the plane drawn at right angles to it through the line, since their common section evidently contains the projections of all points on the line. The projections of parallel straight lines, being the common sections of the plane of projection with a system of planes at right angles thereto, are themselves parallel straight lines.

(ii) If any two parallel straight lines AP and BQ meet a plane in A and B (fig. Art. 89), and if AC and BD be their projections upon it, then by similar triangles

$$AP : AC = BQ : BD,$$

or each of the parallels is in the same ratio to its projection, and in like manner it may be shewn that any segment of either is in the same ratio to its projection. This ratio is the trigonometrical secant of the angle between any of the parallels and its projection.

Hence it is evident that parallel straight lines (and the segments of one and the same straight line) are to one another as their projections upon any plane.

If two or more systems of parallels be projected orthogonally, each line in any system will be in a constant ratio to its projection, but this ratio will change as we pass from one system to another.

PROPOSITION II.

88. *The points of concourse of straight lines or curves correspond to the points of concourse of their projections: tangents project into tangents: and a curve of any degree or class projects into a curve of the same degree or class.*

(i) Since figures are projected by projecting their several points, if two or more figures have a point in common their projections must have the projection of the point in common. Thus if a variable line pass through a *fixed* point, the projection of the line will pass through a fixed point.

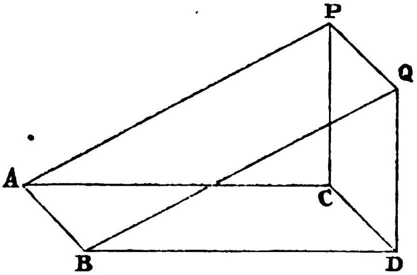
(ii) If P and Q be adjacent points on a curve, and p and q be the feet of the perpendiculars from them to the plane of projection, then if Q be made to coincide with P the perpendicular Qq must coincide with Pp, and the point q with p; in other words, of PQ become the tangent at P, then in the projection pq will become the tangent at p.

(iii) It is hence evident that a curve and any straight line in its plane intersect in the same number of points as their projections, and hence that a curve and its projection must always be of the same degree or order; and further, that the number of tangents which can be drawn from any point to a curve is the same as the number which can be drawn from the projection of the point to the projection of the curve, and hence that a curve and its projection are always of the same class.

PROPOSITION III.

89. *The areas of all figures in one plane are in the same ratio to the areas of their projections on a given plane.*

Let AB be the common section of the primitive plane and the plane of projection, and let PA and QB be drawn from any two points P and Q in the former plane at right angles to AB, and let PC and QD be perpendiculars to the plane of projection. Then since AP and BQ are in a constant ratio to their projections AC and BQ, the rectilinear figure $APQB$ is in the same constant ratio to its projection $ACDB$.



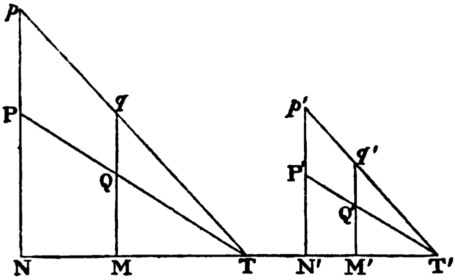
In like manner, if PP' be any segment of AP and QQ' any segment of BQ, the rectilinear area $PP'Q'Q$ is in a constant ratio to its projection; and hence the aggregate of any number of figures as $PP'Q'Q$ is in the same ratio to the aggregate of their projections.

But any rectilinear area in the primitive plane may be divided into elements as $PP'Q'Q$ by drawing planes at right angles to AB, and any curvilinear area may be regarded as the limit of a rectilinear area. Hence every area is to its projection in a constant ratio, which is easily seen to be equal to the secant of the angle between its plane and the plane of its projection.

90. *Plane Projection.*

The above propositions all follow from the constancy of the ratio of AP to its projection, where PA is the perpendicular from any point P in the primitive plane to its common section with the plane of projection. The relations between the locus of P and its projection will evidently still be of the same kind if the former plane be turned about AB into coincidence with the latter, the situation of P in its own plane being supposed to remain undisturbed. The new position of P will lie in AC produced, and will be such that its projection C now divides AP in a constant ratio. Since all points and their projections are thus brought into one plane, we shall speak of the point C which divides AP in a constant ratio as the orthogonal *Plane Projection* of P.

The relations between a figure and its plane projection may of course be obtained directly by plane geometry. Thus if PN be an ordinate to a fixed axis, and if it be divided (externally or internally) in a constant ratio at p, it is easily proved by similar triangles that if the locus of P be a straight line meeting the axis in T, the locus of p will be a straight line meeting the axis in the same point: parallels as PT, $P'T'$ correspond to parallels pT, $p'T'$: and the results of Props. I.—III. in general may be similarly established *in plano*.



It may be proved also that if two straight lines make *supplementary* angles with the axis their projections make supplementary angles with the axis.

The following examples will serve to illustrate the method of orthogonal projection.

91. *The Ellipse.*

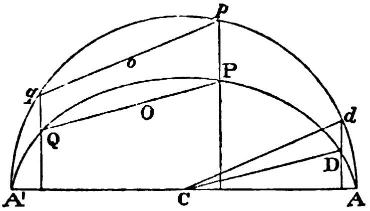
A cylinder being regarded as a cone whose generating lines are parallel, it follows from Art. 72 that the plane sections of a right circular cylinder are ellipses, which may be of any eccentricity. It follows that *an ellipse of any eccentricity may be* *projected orthogonally into a circle* having its plane at right angles to the axis of the cylinder. This circle will evidently be equal to the minor auxiliary circle of the ellipse.

An ellipse may also be projected orthogonally *in plano* into a circle by producing its principal ordinates in the ratio of CA to CB, or by diminishing the ordinates to its minor axis in the ratio of CB to CA; and conversely a circle may be projected *in plano* into an ellipse by increasing or diminishing every ordinate pN in a constant ratio. [Art. 33.

a. Let an ellipse and a circle be projectively related. Let PQ be any one of a system of parallel chords in the ellipse: pq the corresponding chord of the circle, which will itself belong to a system of parallels.

The middle point O of PQ corresponds to the middle point o of pq; and if the locus of the one be a straight line, the locus of the other will be a straight line. But the locus of o in the circle is a straight line, and therefore in the ellipse the locus of the middle points of any system of parallel chords is a straight line. From this it is further evident that conjugate diameters in the ellipse correspond to diameters *at right angles* in the circle.

b. Next let O be a fixed point and PQ any chord of the ellipse passing through it: then pq in the circle will pass through the corresponding fixed point o.



Let CD be the radius of the ellipse parallel to PQ, and Cd the corresponding radius of the circle, which will be parallel to pq. Then by Prop. I.

$$OP : OQ : CD = op : oq : Cd.$$

Therefore $$OP.OQ : CD^2 = op.oq : Cd^2,$$

or ($op.oq$ and Cd in the circle being constant) $OP.OQ$ varies as the square of the parallel semi-diameter CD.

c. Since the ordinates PN and pN in the second figure of Art. 33 are as CB to CA, the area of any segment or sector of the ellipse is to the area of the corresponding segment or sector of the circle as CB to CA, and the area of the whole ellipse is to that of the circle in the same ratio. The area of the ellipse is therefore equal to $\pi.CA.CB$, and it is a mean proportional to the areas of its major and minor auxiliary circles.

d. Any *maximum* or *minimum* or *constant* area related to the ellipse projects into an area similarly related to the circle. Thus, the greatest triangle which can be inscribed in a circle being equilateral, the greatest triangle which can be inscribed in an ellipse may be shewn to be that which has its sides parallel to the tangents at its angular points and its centroid at the centre of the ellipse.

92. *The Equilateral Conics.*

Any hyperbola can be projected into an equilateral hyperbola in the same way that an ellipse can be projected into a circle (Art. 33); and thus from certain properties of the equilateral conics (Art. 61) we can deduce corresponding properties of central conics in general. For example, having proved for the equilateral hyperbola (Art. 65) as well as for the circle, that the conjugate parallelogram is equal to $4CA^2$, we infer that in any central conic the conjugate parallelogram is equal to $AA'.BB'$. In like manner other projective properties which have been established for the two equilateral conics are at once seen to be capable of extension to central conics in general.

93. *Properties of Polars.*

Using the same letters as in Art. 41 but supposing the curve to be a *circle*, we have the angles at o and T right angles, and therefore

$$CO.CT = Co.Ct = CA^2,$$

or T is a fixed point and tT a fixed straight line.

In other words, if a chord of a circle pass through a fixed point O the tangents at its extremities intersect on a fixed straight line tT, and conversely. The same follows at once for the ellipse by orthogonal projection.

94. *Property of a Quadrilateral.*

If $ABCD$ be a fixed quadrilateral inscribed in a circle, PQ a variable diameter of the circle, Pa, Pb, Pc, Pd perpendiculars to AB, CD, AC, BD respectively; then by Euc. VI. C the rectangle $Pa.PQ$ is equal to $PA.PB$, and thus

$$Pa.Pb=\frac{PA.PB.PC.PD}{PQ^2}=Pc.Pd.$$

Project the circle into an ellipse. Then (still using the same letters) $Pa.Pb$ in the ellipse *varies as* $Pc.Pd$, and the lines Pa, Pb, Pc, Pd meet the sides of $ABCD$ at *constant* angles, and therefore vary severally as the perpendiculars from P to those sides.

The theorem is inferred to be true for the hyperbola regarded as a quasi-ellipse (Scholium C, p. 101), as also for the parabola. Hence *the products of the perpendiculars from any point on a conic to the sides of a fixed inscribed quadrilateral, taken in opposite pairs, are in a constant ratio;* and conversely, if a point be thus related to a fixed quadrilateral, its locus will be a conic circumscribing the quadrilateral.

It follows by supposing two opposite sides of the quadrilateral to vanish that *the product of the perpendiculars to any two fixed tangents to a conic from a variable point on the curve varies as the square of the perpendicular from that point to their chord of contact.*

95. *Curvature.*

The common chord of an ellipse and its circle of curvature at any point and their common tangent at that point make supplementary angles with the axis (Art. 85), and may therefore be projected into a chord and tangent at a point of a circle making supplementary angles with a given diameter.

[Art. 90.

Given a point O on a circle, three positions may be found on the curve of the point P such that OP and the tangent at P make supplementary angles with a given diameter, and the three positions of P determine an equilateral triangle.

Hence it may be deduced that there are three points on an ellipse, lying at the vertices of a maximum inscribed triangle, whose osculating circles cointersect at a given point on the circumference of the ellipse. [Ex. 650.

PARALLEL PROJECTION.

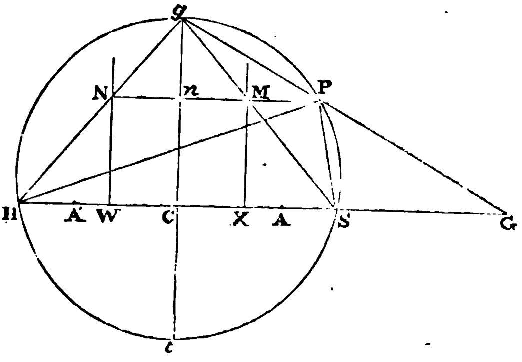
96. The parallel projections of straight lines, being the common sections of parallel planes drawn through them with the plane of projection, are themselves parallel straight lines; and it is easily deduced that Props. I.–III. in general are applicable to parallel projection. Moreover it is to be noted that in the figure of Art. 90 the ordinate PN may be supposed to meet the axis at *any constant angle* and not necessarily at right angles. We may therefore "project" a point obliquely also *in plano* by increasing or diminishing its ordinate (drawn in any given direction) in a constant ratio.

Although parallel projection is not essentially more general than orthogonal projection, it enables us to effect some constructions with greater ease and directness, as will appear from the next article.

97. *Any two angles of a triangle may be projected into angles of given magnitude.*

(i) Given the angles SHP and HSP, it is required to project them into angles equal to SHg and HSg respectively. This may be effected most simply by parallel projection (*in plano* or in space) as follows.

Take SH as the axis of projection, and let it meet gP in G. Then (1) if $gG : PG$ be taken as the projecting ratio, the triangle SPH may be projected *in plano* by ordinates parallel to PG into the triangle SgH; or (2) if the triangles be situated in different planes (having SH for their common section), the one may be projected into the other by lines parallel to Pg.



It is hence evident that *any triangle may be projected by parallel lines into an equilateral triangle;* and in like manner that any parallelogram may be projected into a square.

(ii) To project any triangle *SPH orthogonally* into an equilateral triangle, let an ellipse be described about it so as to have its centre at the centroid of the triangle. Then if the ellipse be projected into a circle, *SPH* will at the same time project into an equilateral triangle inscribed in the circle, since each side of the projected triangle will be parallel to the tangent at the opposite vertex.

LAMBERT'S THEOREM.[*]

98. *The intercept on any focal vector of an ellipse between the curve and any chord parallel to the tangent at its extremity is equal to the diametral sagitta of the arc cut off by the corresponding chord in the auxiliary circle.*

In an ellipse take any focal vector *SP*, let the diameter conjugate to *CP* meet *SP* in *K*, and let any chord *QQ'* parallel to *CK* meet *SP* in *O*.

Produce the ordinates of *P*, *Q*, *Q'* outwards to meet the auxiliary circle in *p*, *q*, *q'*, and let *m* and *o* be the middle points of *QQ'* and *qq'*, which will lie on a common ordinate *omM*.

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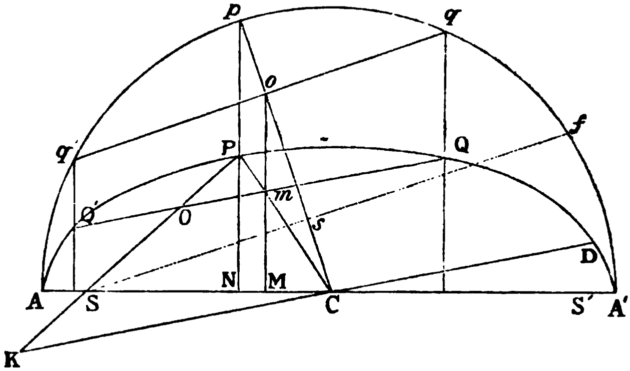
[*] This name is given to the theorem in elliptic motion deduced below in the Scholium from Art. 99.

Then $PO : PK = Pm : PC = po : pC$



by parallels; and PK is equal to pC (Art. 38, Cor. 3); and therefore PO and po are equal, as was to be proved.

If O' and o' be any second positions of O and o, it follows that OO' is equal to oo': thus by taking O' at S we infer that *SO is equal to the altitude of the triangle Sqq'.* If the parallel from S to qq' meet Cp in s, it is thus evident that every segment of the line $KSOP$ is equal to the corresponding segment of the radius $Csop$. Hence

$$KS.KO = Cs.Co = CS.CM,$$

since the angles at s and M are right angles.

99. *If in two ellipses described on equal major axes there be taken equal chords QQ' such that $SQ+SQ'$ is of the same magnitude in both ellipses, the areas of the two focal sectors SQQ' will be in the subduplicate ratio of the latera recta of their ellipses.*

In each of two ellipses having equal major axes make the same construction as in Art. 98, with the proviso that SP[*] and PO shall be of the same lengths in both ellipses; then will QQ' and $SQ+SQ'$ be of the same lengths in both ellipses, and the areas SQQ' will be in the subduplicate ratio of the latera recta of their ellipses, and conversely.

(i) For the semi-diameter CD parallel to QQ', being a mean proportional to SP and $AA'-SP$, is of the same length in both ellipses; and, the sagittæ po in the two circles being equal (Art. 98), the chords qq' are equal; whence it follows in virtue of the projective relation

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[*] Take SP at random in the less eccentric ellipse, and find an equal SP in the other.

$$QQ' : CD = qq' : CA,$$

that the chords QQ' are equal in the two ellipses.

(ii) And since CM is the abscissa of the middle point of QQ', therefore

$$SQ + SQ' = AA' - 2\frac{CS}{CA}.CM,^*$$

where $CS.CM$, or $KS.KO$ (Art. 98), is the same in both ellipses; and therefore also $SQ + SQ'$ is of the same magnitude in both.

(iii) It is further evident that the segments qpq' are equal and the triangles Sqq' are equal in both circles, and hence that the two sectors Sqq' are equal. It follows by projection that (the major axes of the two ellipses being equal) the focal sectors SQQ' are in the ratio of the minor axes of their ellipses, or in the subduplicate ratio of their latera recta. Conversely, if two focal sectors SQQ' of any pair of ellipses described on equal major axes be thus related, then will QQ' and $SQ + SQ'$ be of the same lengths in both ellipses.

100. *The area of any focal sector SQQ' of a central conic divided by the square root of its latus rectum may be expressed in terms of QQ', $SQ + SQ'$, and the tranverse axis, and is independent of the magnitude of the conjugate axis.*

This may be inferred from Art. 99 for the ellipse: or it may be proved by the following method, which will be seen to be applicable to the hyperbola as well as to the ellipse.

In an ellipse whose axes are AA' and BB' take any diameter PCP', and let PS meet the conjugate diameter CD in K and the tangent at P' in H, so that PK is equal to CA and PH to AA'.

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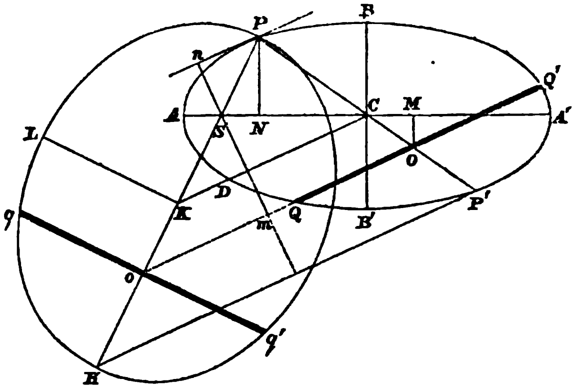
* Assume that at any point P on the curve $SP = CA \pm e.CN$. Notice also in Art. 100 (ii) that $SK = PK - SP = CA - SP = e.CN$.



With S as focus and P and H as vertices describe a second ellipse, and let KL be its semi-axis conjugate. Then if S' be the further focus of the first ellipse,

$$CD^2 = SP.PS' = SP.SH = KL^2.$$

(i) Let QOQ' be any double ordinate to PP', let it meet PH in o, and let qoq' be a double ordinate to PH in the second ellipse. Then by the nature of ordinates,

$$\begin{aligned} QO^2 : CD^2 = PO.OP' : CP^2 &= Po.oH : KP^2 \\ &= \quad qo^2 \quad : KL^2, \end{aligned}$$

or QO is equal to qo and QQ' to qq'.

(ii) And since $SK : CN = CS : CA = CS : PK$,

and $CN : CM = CP : CO = PK : Ko$;

therefore $SK : CM = CS : Ko$,

or in terms of the eccentricities e and e',

$$e.CM = e'.Ko;$$

and therefore

$$\begin{aligned} SQ + SQ' = AA' + 2e.CM &= PH + 2e'.Ko \\ &= Sq + Sq'. \end{aligned}$$

(iii) Let a line through S at right angles to QQ' meet QQ' in m and the tangent at P in n; then will the equal chords QQ' and qq' in any two consecutive positions cut off elements of area which, being as their own breadths, are in the *constant* *ratio* of Sm to So; and this is also the ratio of the triangles SQQ' and Sqq' on equal bases.

Therefore the whole segments $QP'Q'$ and qHq', as also the sectors SQQ' and Sqq', are as Sm to So, or as Sn to SP, or as CB to CD or KL; that is to say (the transverse axes being equal) the sectors SQQ' and Sqq' are in the subduplicate ratio of the latera recta of their ellipses,

Lastly (calling the second ellipse described as above the *auxiliary ellipse*) if any two ellipses having equal major axes be placed so as to have a common focal vector SP, they will have the same auxiliary ellipse; and it is inferred by comparing each of the two ellipses with their auxiliary ellipse that their corresponding focal sectors SQQ' will be in the subduplicate ratio of their latera recta provided that QQ' and $SQ + SQ'$ be equal in both. Hence, when the three lengths QQ', $SQ + SQ'$, AA' are given the area of the focal sector SQQ' divided by the square root of the latus rectum is determined.

SCHOLIUM.

The substance of the preceding section may be found in LAMBERT'S *Insigniores Orbitæ Cometarum Proprietates*[*] (pp. 102–125), where it is deduced that *the time in any arc QQ' of an elliptic orbit described about S is a function of $SQ + SQ'$, the chord QQ', and the major axis*, and does not depend upon the minor axis. For an investigation of the expression for the time the reader may consult the *Messenger of Mathematics*, vol. VII. p. 97: the result may also be obtained in the same form by the following process, which has the advantage of giving a more direct geometrical interpretation to some of the symbols employed.

Assuming the results of Arts. 98, 99, and noticing further that sf is equal to CD (Ex. 651), let α denote the angle oCq, and β the angle sCf. Then

$$\cos\alpha \,.\, \cos\beta = \frac{Co\,.\,Cs}{CA^2} = \frac{CM\,.\,CS}{CA^2} = 1 - \frac{SQ + SQ'}{AA'},$$

$$\sin\alpha \,.\, \sin\beta = \frac{\frac{1}{2}qq'\,.\,sf}{CA^2} = \frac{\frac{1}{2}qq'\,.\,CD}{CA^2} = \frac{QQ'}{AA'};$$

and therefore $$1 - \cos(\beta \pm \alpha) = \frac{SQ + SQ' \pm QQ'}{AA'}.$$

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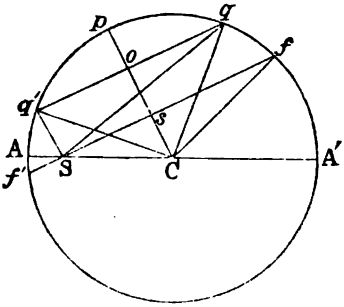
[*] See above, p. 57; and for a further account of the work see the *Messenger of Mathematics*, vol. IX. p. 63.

Let n denote the mean motion in the ellipse: t the time in the arc QQ', which is proportional to the sectorial area SQQ' (or $\frac{CB}{CA}$ sect.Sqq') divided by the square root of the latus rectum (NEWTON's *Principia*, Lib. I. Sect. III. prop. 14, theor. 6). Then since in the circle



$$\text{sect.}\, Sqq' = \text{sect.}\, Cqq' - \triangle\,(Cqq' - Sqq') = CA^2\,(\alpha - \sin\alpha\,.\cos\beta),$$

it may be deduced that

$$nt = \{\beta + \alpha - \sin(\beta + \alpha)\} - \{\beta - \alpha - \sin(\beta - \alpha)\},$$

where $\beta + \alpha$ and $\beta - \alpha$ are known functions of AA', QQ', and $SQ + SQ'$.

The corresponding theorem in parabolic motion had been given by EULER in the *Miscellanea Berolinensia*, tom. VII. pp. 19, 20 (1743). LAMBERT (1761) extended it to the central conics generally; although Lexell in his elaborate article (*Nova Acta Academiæ Scientiarum Imperialis Petropolitanæ*, tom. I. pp. (140)—(183), 1787) overlooks Lambert's references in Preface and text (§§ 57, 95, 213, &c.) to the hyperbola. On the parabola see Examples 691 and 692.

EXAMPLES.

651. The diameter parallel to any focal chord of an ellipse is equal to the projectively corresponding chord of its major auxiliary circle.

652. The orthogonal or parallel projection of a parabola is a parabola; and an ellipse or hyperbola may be projected into an ellipse or hyperbola of any eccentricity.

653. Shew how to project a parabola and a given point within it so that the projection of the point may be the focus

of the projection of the curve:* and thus deduce (from a property of the directrix) that the tangents to a parabola at the extremities of any chord drawn through a fixed point within it intersect upon a fixed straight line.

654. Any two similar and coaxal ellipses may be projected into concentric circles. Hence shew that a chord of an ellipse which always touches a similar and coaxal ellipse is bisected at its point of contact, and that it cuts off a constant area from the outer ellipse; and shew that the portions of any chord intercepted between the two curves are equal.

655. Shew how to project any two coaxal ellipses or hyperbolas into confocal ellipses or hyperbolas; and extend the theorem of Ex. 382 to any pair of coaxal conics of the same species.

656. If a chord of an ellipse and the tangents at its extremities contain a constant area, the chord cuts off a constant area from the ellipse and touches a similar ellipse, and the tangents at its extremities intersect on another similar ellipse.

657. If CP and CD be conjugate radii of an ellipse, and if the lines connecting P and D with opposite ends of the major axis intersect in O; then will DOP and an extremity of the minor axis determine a parallelogram. When is its area least or greatest?

658. The least triangle circumscribing a given ellipse has its sides bisected at the points of contact.

659. The greatest ellipse which can be inscribed in a given parallelogram is that which bisects its sides.

660. An n-gon described about an ellipse so as to have its sides bisected at their points of contact is of constant area, and the n-gon formed by joining every two successive points of contact is of constant area.

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* Let TP and TQ be the tangents whose chord of contact PQ passes through and is bisected by the given point; and let TPQ be projected into an isosceles triangle right-angled at the projection of T (Art. 97).

661. If a triangle be inscribed in an ellipse, the parallels through its vertices to the diameters bisecting the opposite sides meet in a point.

662. Parallel chords drawn to an ellipse through the extremities of conjugate diameters meet the curve again at the extremities of conjugate diameters.

663. Two ellipses of equal eccentricities and parallel major axes can have only two points in common; and the common chords of three such ellipses meet in a point.

664. P, Q, R being three points on an ellipse, a diameter ACA' bisects PQ and meets RP in N and RQ in T; prove that $CN.CT = CA^2$.

665. Through the centre of an ellipse and the points of concourse and contact of any two tangents a similar and similarly situated ellipse can be drawn.

666. A, B, C are similar and similarly situated ellipses, and B is concentric with A and passes through the centre of C. Shew that the common chord of A and C is parallel to the tangent to B at the centre of C.

667. Through a given internal point draw a straight line cutting off a minimum area from a given ellipse.

668. If the tangent at the vertex A of an ellipse meets a similar coaxal ellipse in T and T', any chord of the former drawn from A is equal to half the sum or difference of the parallel chords of the latter through T and T'.

669. Determine the greatest triangle that can be inscribed in an ellipse with one angular point at a given point on the circumference.

670. Given an ellipse, if a parallelogram be inscribed in it so that one side divides the diameter conjugate thereto in a constant ratio; or if a parallelogram be described about it so that one of its diagonals is divided in a constant ratio by the curve; the area of the parallelogram will be constant.

671. The tangents to an ellipse at P and P' are parallel, any two conjugate diameters meet them in D and D', and any third tangent meets them in T and T'; shew that

$$PD : PT = P'T' : P'D'.$$

672. A triangle ABC inscribed in an ellipse has its centroid at the centre of the ellipse; shew that the tangents at the opposite extremities of the diameters through A, B, C form a triangle similar to and four times as great as triangle ABC.

673. If two conjugate hyperbolas having a pair of conjugate diameters of an ellipse for asymptotes cut the ellipse at points lying on four diameters 1, 2, 3, 4 taken in order: then will 1, 3 and 2, 4 be conjugate in the ellipse, and 1, 4 and 2, 3 in the hyperbolas.

674. The locus of the middle point of a chord of an ellipse drawn through a fixed point is a similar ellipse, having its centre midway between the fixed point and the centre of the given ellipse.

675. Determine the greatest isosceles triangle that can be inscribed in an ellipse with its base parallel to either axis.

676. Any two radii of a circle and any pair of the radii at right angles thereto determine equal triangles: what is the corresponding property of the ellipse?

677. Any double ordinate to a given diameter of an ellipse being divided into segments whose product is constant, the point of section traces a similar coaxal ellipse.

678. If the tangents at T and T' to an ellipse meet on a similar coaxal ellipse having AA' for major axis, shew that the loci of the points $(AT, A'T')$ and $(AT', A'T)$ are ellipses similar to the former.

679. If PP' be a diameter of an ellipse and Q, R be any two points on the curve, shew that the line joining the intersections of PQ, $P'R$ and PR, $P'Q$ passes through the intersection of the tangents at Q and R and is conjugate in direction to PP'.

680. In a given ellipse shew how to place a chord passing through a given point in the axis which shall contain with the radii to its extremities a maximum triangle.

681. From the extremities of any diameter of an ellipse straight lines are drawn to either focus; shew that the radii parallel thereto in the auxiliary circle meet the circle at points lying on a chord through the focus.

682. A triangle of constant area being inscribed in an ellipse, find the locus of its centroid.

683. If P and Q be projectively corresponding points on an ellipse and its major auxiliary circle, the radius of the ellipse in the direction CQ is equal to the perpendicular from C to the tangent at P.

684. AA' being a diameter of a conic and Q any point on the tangent at A, shew that the line $A'Q$ cuts the conic in a point B such that the tangent thereat bisects AQ.

685. If CP, CP' be semi-diameters of an ellipse, Q and Q' the points in which the conjugate diameters meet the tangents at P' and P respectively; the triangle determined by a pair of its semi-axes is a mean proportional to the triangles CPP' and CQQ'.

686. If a triangle inscribed in an ellipse have its sides parallel to the diameters b', b'', b''' and the focal chords c', c'', c'''; and if a, b be the axes and p the parameter of the ellipse, and D the diameter of the circle circumscribing the triangle; shew that

$$D = \frac{b'b''b'''^{*}}{ab} = \left(\frac{c'c''c'''}{p}\right)^{\frac{1}{2}}.$$

687. The square root of the continued product of the six focal chords of an ellipse parallel to its six chords of intersection with a circle is equal to the parameter of the ellipse multiplied by the square of the diameter of the circle.

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* The theorem is due to **MAC CULLAGH**. (Salmon's *Conic Sections*, Art. 369).

688. Shew that $S'B$ (fig. p. 79) is perpendicular to the tangent from X to the auxiliary circle. If Z be the point of contact of this tangent and G its intersection with $S'B$, determine the eccentricity of the ellipse so that the lines CZ, SG, XB may meet in a point.

689. Two particles describe the same ellipse subject to the same force at the centre: shew that their directions of motion at any time intersect on a similar ellipse.

690. A given triangle may be orthogonally projected from an equilateral triangle; or it may be orthogonally projected into an equilateral triangle. Determine by a geometrical construction the magnitudes of these equilateral triangles.

691. Prove by the method of Art. 100 that the area of any focal sector SMN of a parabola divided by the square root of its latus rectum may be expressed in terms of $SM + SN$ and MN.

692. Prove also that if PV be the diametral sagitta of the arc MN, then

$$\frac{SM + SN \pm MN}{2} = \{\sqrt{(SP)} \pm \sqrt{(PV)}\}^2;$$

and from this last result deduce that

$$\frac{3}{\sqrt{(AS)}} \text{ sector } SMN = \left\{\frac{SM + SN + MN}{2}\right\}^{\frac{3}{2}} - \left\{\frac{SM + SN - MN}{2}\right\}^{\frac{3}{2}}.^*$$

693. If in any two ellipses or hyperbolas on equal transverse axes there be taken equal focal vectors SP, and if chords QQ' be drawn in them parallel to the tangents at P and so as to make equal intercepts on the equal focal vectors; shew that the chords thus drawn are equal in the two conics, and that $SQ + SQ'$ is of the same length in both, and that the

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[*] This is here suggested as perhaps the most direct way of establishing geometrically the theorem in parabolic motion commonly ascribed to LAMBERT but previously discovered by EULER (Scholium, p. 242).

focal sectors SQQ' are in the subduplicate ratio of the latera recta of their ellipses or hyperbolas.*

694. Given, in a conic orbit described about the focus S under the action of a given force, a chord QQ', the parallel focal chord, and the sum of the terminal focal vectors $SQ+SQ'$; the time of describing the arc QQ' is constant.†

695. Given the directions of two sides of a triangle inscribed in a given ellipse, determine the envelope of its third side.

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* The proof is very much as in Art. 100; but the two conics are thus compared directly and without the interposition of an auxiliary conic in which the chord qq' of the sector is at right angles to the principal axis.

† This extended enunciation of LAMBERT'S theorem (as I am informed by Prof. TOWNSEND) was given in a lecture in the year 1845 by the late Prof. MAC CULLAGH of Dublin, who proved that the sectorial area $SQPQ'$ and the square root of the latus rectum of the orbit vary severally as the sine of the angle between SP and QQ', having previously shewn that OP is constant, and that every chord parallel to QQ' which cuts SP in a constant ratio is of constant length.

CHAPTER X.

CROSS RATIO AND INVOLUTION.

101. A SET of points on a straight line constitute a *Range*[*] or row, of which the line is called the base or axis; and a set of straight lines radiating from a point constitute a *Pencil*, of which the point is called the *Vertex* or the centre and the lines are called *Rays* or radiants.

The ratio of the ratios in which the straight line joining two points is divided by any other two points in its length is called a *Double Section Ratio*, or an *Anharmonic Ratio*, or a *Cross Ratio* of the four points. If two ranges of four points have one cross ratio of the one equal to one cross ratio of the other, it may be shewn that every cross ratio of the one is equal to the corresponding cross ratio of the other: we may therefore in practice speak of a tetrad as having one cross ratio only, although its four points determine six segments to each of which belong a reciprocal pair of cross ratios.[†]

102. If a line BD be divided at A and C, the ratio $AB.CD : AD.CB$ (which is a *cross-multiple* of the ratios $\frac{AB}{AD}$ and $\frac{CB}{CD}$) may be taken as the cross ratio of the points $ABCD$, and it is expressed by the notation $\{ABCD\}$. It may

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[*] Collinear points are (or were) sometimes said to "range on a straight line"—for an example see Booth's *New Geometrical Methods*, vol. I. §48.

[†] For a further exposition of the principles of Anharmonic Section see TOWNSEND's *Modern Geometry of the Point Line and Circle*, vol. II. The purpose of this chapter is not so much to establish these principles as to apply them to conics.

be shewn that *this ratio is unaltered by the interchange of any two points of the range provided that the remaining two points be at the same time interchanged.* In other words, it may be shewn that

$$\{ABCD\} = \{BADC\} = \{CDAB\} = \{DCBA\}.$$

The cross ratio of a pencil of four rays is that of the range which it determines on any transversal: this ratio will be shewn to be *independent of the position of the transversal* (Art. 103). The cross ratio of four radiants $ABCD$ is expressed by $\{ABCD\}$, and the cross ratio of four radiants Oa, Ob, Oc, Od by $O\,\{abcd\}$ or $\{O.abcd\}$.

Tetrads of points and rays are said to be *equal* or *equianharmonic* when their cross ratios are equal; and ranges and pencils of any number of constituents are said to be *homographic** when every tetrad of the one system and the corresponding tetrad of the other have equal cross ratios. The notation

$$\{ABCDE \text{ \&c.}\} = \{A'B'C'D'E' \text{ \&c.}\}$$

is used to express that two systems whose constituents correspond in pairs are homographic. More briefly, if A and A' be a variable pair of homologous (*i.e.* corresponding) constituents of two homographic systems, the equation $\{A\} = \{A'\}$ may be used to signify that every four positions of A and the corresponding positions of A' have equal cross ratios. The corresponding notation for a pencil having its vertex at O is $O\,\{A\}$.

Thus far we have spoken only of collinear points and of concurrent lines, but the following definitions may be added in anticipation of Art. 113. The cross ratio of *Four Points on a Conic* is that of the pencil which they subtend at any fifth point on the curve; and the cross ratio of *Four tangents to a Conic* is that of the range which they determine on any fifth tangent thereto. In the special case of the *Circle* it is at once evident

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* This term is sometimes confined to systems of more than four constituents, but it may be used with equal fitness for the special cause of tetrads, in place of the longer word *equianharmonic*. The still shorter term *equal* (although not in general so used) may often be employed with advantage as an abbreviation for "equal as regards cross ratio"—for which the term *equicross* is suggested by Prof. TOWNSEND.

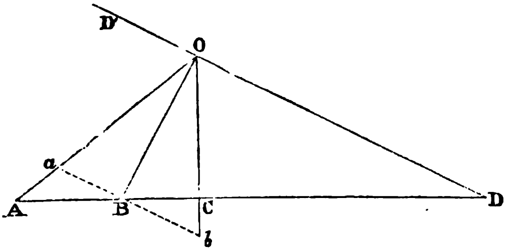
that these ratios are *independent of the position of the fifth point or tangent*, since four given points on a circle subtend a pencil of *constant angles* at any fifth point on the curve, and since the segments of any fifth tangent by four fixed tangents subtend *constant angles* at the centre. It will be shewn in Art. 113 that the cross ratios of four given points on (or tangents to) any conic are likewise constant.

We proceed to establish certain Lemmas relating mainly (1) to Cross Ratio in general, and (2) to the special case of homography called Involution, with a view to their eventual application to some of the fundamental properties of conics.

CROSS RATIO.

103. *The cross ratio of the points in which four fixed radiants are met by a variable transversal is constant.*[*]

(i) Let four lines radiating from O be met by any transversal in the points A, B, C, D, and let the parallel through B to OD meet OA in a and OC in b.



Then by similar triangles,

$$AB : AD = aB : DO$$

and

$$CD : CB = DO : Bb.$$

Hence

$$\{ABCD\} = \frac{AB.CD}{AD.CB} = \frac{aB}{bB},$$

which is constant for all positions of ab parallel to OD; and therefore $\{ABCD\}$ is constant for all positions of $ABCD$ without restriction.

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[*] It is to be noted that this very important theorem of the ancients implicitly contains the method of Conical Projection.

We have thus proved that

$$\{ABCD\} = \{aBb\infty\},$$

where ∞ denotes the point at infinity in which aBb meets OD.

It is to be noted that the transversal may be drawn so as to cut the radiants (produced through O) upon either side of O. Thus in the figure of Art. 107 (i), considering the pencil whose vertex is O and supposing QO to meet AB in E and CD in F, we have

$$\{DMAQ\} = \{BPAE\} = \{BRCQ\} = \{DPCF\},$$

the rays of the pencil being taken in the same order in every case.

(ii) It is further evident that different pencils have equal cross ratios when their corresponding angles are equal or supplementary.

For example, if SR and ST turn about S (Art. 9, Cor. 1) so as to be always at right angles, the pencils $S\{R\}$ and $S\{T\}$ are equiangular and therefore homographic. Or again, if TS and TH in Art. 50 remain fixed whilst the conic varies (so that the tangents from T always make equal or supplementary angles with TS and TH respectively), the pencils $T\{P\}$ and $T\{Q\}$ are homographic.

104. *Condition that a variable straight line should pass through a fixed point.*

Four rays determine equal ranges upon any two transversals (Art. 103); and conversely, if two equal tetrads of points on different axes be such that three of the lines joining them in homologous pairs meet in a point, the join* of the fourth pair must pass through the same point.

In particular, if A be a point on one of the four rays, they determine equal ranges $\{ABCD\}$ and $\{AB'C'D'\}$ on any two transversals through A.

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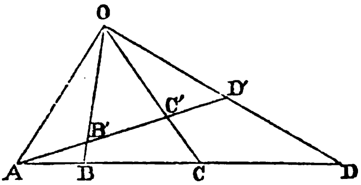
*. This term is used by some writers as an abbreviation for the *line joining* two points. See for example HENRICI'S *Elementary Geometry, Congruent Figures*, § 47 (London, 1879). The "join" of two lines is their common point.



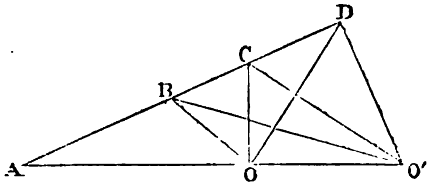
Conversely, if BB' and CC' be any two assumed positions of a variable straight line, O their point of concourse and A that of BC and $B'C'$, and if OA be a third position and DD' any other position of the variable line, the condition that DD' should pass through the fixed point O is that $\{ABCD\}$ should be always equal to $\{AB'C'D'\}$.

105. *Condition that a variable point should lie on a fixed straight line.*

A range of four points subtends equal pencils at any two vertices; and conversely, if two equal pencils having different vertices be such that three pairs of their corresponding rays intersect on one straight line, the fourth pair must intersect on the same straight line.

In particular, if O and O' be the vertices of two pencils which have $OO'A$ for a common ray, and if B, C, D be the intersections of their remaining three pairs of rays, then will BCD be a straight line provided that

$$O\{ABCD\} = O'\{ABCD\}.$$



If now (supposing the rays of the two pencils to be variable) the intersections of two pairs of homologous rays be at given points B and C, the remaining intersection D will lie on the fixed straight line BC.

This result and that of Art. 104 may be together expressed as follows:

If two tetrads of points or rays $ABCD$ and $A'B'C'D'$ be equianharmonic, then if one pair (A, A') coalesce the remaining three pairs (B, B'), (C, C'), (D, D') are in perspective.

106. *Condition that a tetrad of points or rays may be harmonic.*

(i) If BD be divided harmonically at A and C, and if the signs ± be used to denote opposite directions, then it is easily seen that

$$\frac{AB}{AD}\cdot\frac{CD}{CB}=\frac{AD}{AB}\cdot\frac{CB}{CD}=\frac{CB}{CD}\cdot\frac{AD}{AB}=-1,$$

or $$\{ABCD\}=\{ADCB\}=\{CBAD\}=-1.$$

The condition that the range $ABCD$ should be harmonic is therefore that the points A and C or the points B and D should be *separately* interchangeable. Each pair of interchangeable points are said to be *harmonically conjugate* to each other with respect to the remaining pair.

(ii) A pencil $O\{ABCD\}$ will be harmonic if a transversal drawn parallel to one of its rays OD be cut in *equal* and opposite segments Ba and Bb (fig. Art. 103) by the remaining three rays. For example, since DP in Art. 60 is parallel to one asymptote and is bisected by the other, it follows that *the asymptotes and any two conjugate diameters of a hyperbola constitute a harmonic pencil.*

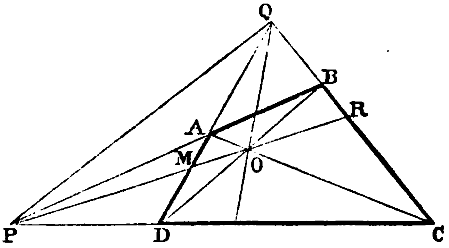
(iii) A pencil will also be harmonic if two of its rays be the bisectors of the angles between the other two. For example, *the tangent and normal and the focal distances at any point of a conic constitute a harmonic pencil.*

107. *Properties of a complete quadrilateral.*

Let $ABCD$ be a quadrilateral, and let O, P, Q be the intersections of (AC, BD), (AB, CD), (AD, BC) respectively. Then the figure as thus completed is called a Complete Quadrilateral, and PQ is called its third Diagonal.

(i) Let the ray PO meet AD in M and BC in R. Then by the properties of radiants (Art. 103),

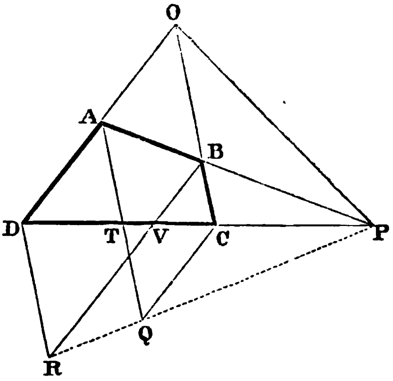
$$O\{DMAQ\} = O\{BRCQ\} = P\{BRCQ\} = P\{AMDQ\};$$



and therefore the range $\{DMAQ\}$, in which the conjugate points A and D have been shewn to be interchangeable, is harmonic. [Art. 106 (i).

Thus it appears that the two pencils whose vertices are O and P are harmonic, and in like manner that the pencil Q is harmonic; and hence that *each of the three diagonals of the complete quadrilateral is divided harmonically by its remaining two diagonals.*

(ii) Let the sides AB, DC of a quadrilateral meet in P and the sides DA, CB in O. Complete the parallelograms $BODR$ and $AOCQ$, and let BR, PD meet in V and AQ, PD in T.



Then by parallels

$$PC : CT = PB : BA = PV : VD,$$

or

$$PC : PV = CT : VD = CQ : VR.$$

Hence PQR is a straight line, and therefore the middle points of OP, OQ, OR lie on a straight line parallel to PQR.

But the middle point of OQ is also the middle point of AC, since the diagonals of parallelograms bisect one another; and for the same reason the middle point of OR is also the middle point of BD.

Therefore *the middle points of the three diagonals AC, BD, OP of a complete quadrilateral lie on a straight line.*

The straight line bisecting the three diagonals of a complete quadrilateral may be called the DIAMETER of the quadrilateral (Ex. 375, note).

SCHOLIUM A.

The fundamental anharmonic property of a pencil of four rays constitutes prop. 129 of the seventh book of the Συναγωγή of PAPPUS (vol. II. p. 871, ed. Hultsch), which affirms that *if two straight lines $ABCD$ and $AB'C'D'$ be drawn across three radiants OB, OC, OD* (fig. Art. 104), *then*

$$AB \,.\, DC : AD \,.\, CB = AB' \,.\, D'C' : AD' \,.\, B'C'.$$

This property is introduced (pp. 867—919) under the head of Lemmas on EUCLID'S three books of Porisms, a subject treated at length by ROB. SIMSON in the work *De Porismatibus* contained in his *Opera quædam reliqua* (Glasguæ, 1776), where some things that were obscure in Pappus are made clear. In the same section of the Συναγωγή (prop. 131) is contained in substance the theorem of Art. 107, that *either diagonal of a quadrilateral is cut harmonically by its other diagonal and the join of the joins* [note, p. 252] *of its opposite sides* (Chasles *Aperçu Historique* p. 36, 1875). And further, it will appear (Schol. C) that a theorem in Pappus implicitly contains the fundamental anharmonic property of four points on a conic.

The ancient theory of transversals was revived and largely developed in the first half of the seventeenth century by DESARGUES, to whom PONCELET, in the introduction to his *Traité des Propriétés Projectives des Figures* (Paris, 1822), applies the title of honour "le MONGE de son siècle" (p. xxxviii). For some further bibliographical information on the subject see Poncelet *loc. cit.* and Chasles *Aperçu Historique*. Of CARNOT'S *Géométrie de Position* (Paris, 1803) Poncelet remarks (p. xliv) that in that work "se trouve exposée pour la première fois et dans toute sa généralité cette belle Théorie des transversales dont nous avons déjà si souvent parlé dans ce qui précède, et dont les Anciens n'avaient fait qu'entrevoir les principes et la fécondité." This method has been still further developed by CHASLES "qui a mis en œuvre d'une manière très-heureuse les principes de la Théorie des transversales pour démontrer la plupart des théorèmes de MONGE" (ibid. p. xlv). A good section on Transversals was contained in the 12th edition of Hutton's *Course of Mathematics*, vol. II. 214–46 (London, 1843).

For the cross ratio—or as he called it "Doppelschnittsverhältniss" (*ratio bisectionalis*)*—of four points MOEBIUS introduced the notation (A, B, C, D) in his celebrated *Barycentrische Calcul*, § 183, p. 246 Leipzig, 1827); and he used the notation (c, d, b, a) for the cross ratio of the range determined by four lines c, d, b, a upon any transversal e (p. 256). Chasles (*Traité des Sections Coniques* Pt. I. p. 7, 1865), following the example of Möbius, adopts the notation (a, b, c, d) for a point-tetrad, and uses $O\,(a, b, c, d)$, an abbreviation by separation of symbols for (Oa, Ob, Oc, Od), to denote the anharmonic ratio of a tetrad of radiants. More generally, whether the letters denote points or lines, we may write $O\{abcd\}$ for the cross ratio of the "joins" of (O, a), (O, b), (O, c), (O, d). The form $\{O\,.\,abcd\}$, used in SALMON's *Conic Sections*, is a convenient substitute for $O\,\{abcd\}$ in cases in which cross ratios are compounded with one another (Möbius, p. 257).

The term *Anharmonic* was invented by Chasles and has been widely used; but it is an artificial word and wanting in brevity, and perhaps ought rather to mean non-harmonic. The short and significant term *Cross Ratio* (which expresses the result of compounding two simple ratios crosswise) was coined by the late Prof. CLIFFORD, and may be found in his *Elements of Dynamics* (p. 42) published some four or five years later (London, 1878). It may be used also with a secondary reference to the transversal or cross-line on which the ratio is estimated.

INVOLUTION.

108. If three or more pairs of points A, A'; B, B'; C, C'; &c. be taken on a straight line at such distances from a point O thereon that

$$OA.OA' = OB.OB' = OC.OC' = \&c.,$$

they are said to constitute a system in *Involution*. The point O is called the *Centre* and the points (A, A'), (B, B'), (C, C'), &c. are called *Conjugate Points* or *Couples* of the involution. The centre is evidently conjugate to the point at infinity upon the axis.

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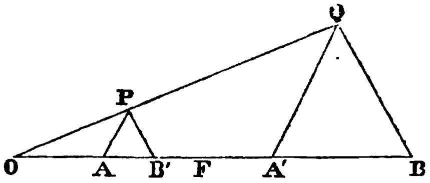
* STEINER called it "Doppelverhältniss."

If the several points of an involution lie on the same side of the centre as their conjugates respectively, the products $OA.OA'$, &c. are to be regarded as positive, and the involution is said to be *positive;* but if conjugate points lie on opposite sides of the centre the involution is said to be *negative.* In a positive involution there are two points F and F' (on opposite sides of the centre) each of which coincides with its own conjugate: these are called the *Double Points* or *Foci* of the involution. In a negative involution the foci are imaginary.

When two pairs of conjugate points A, B and A', B' are given the involution is completely determined. For if two pairs of parallels be drawn through A, A' and B, B' respectively, and if they intersect in P and Q, the line PQ evidently meets the axis of the involution in a fixed point O such that

$$OA : OA' = OP : OQ = OB' : OB,$$

or
$$OA.OB = OA'.OB'.$$



Thus the centre O and the constant of the involution are known, and the system is completely determined.*

A *Pencil* of lines in involution is a pencil which determines a row of points in involution on any transversal: every such pencil has one pair of *Double Rays*, on which lie the double points of all its transversals. [Art. 112.

109. *A system of circles having a common radical axis determine pairs of points in involution on any transversal.*

For if a system of circles be drawn through two points P

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* The six lines joining any four points $PQP'Q'$ determine an involution on any transversal: the construction in the text results from taking P' and Q' at infinity.

and Q, and if a transversal meet PQ in O and meet any two of the circles in A, A' and B, B', it is evident that

$$OA.OA' = OB.OB' = OP.OQ.$$

Conversely, if circles be drawn through the several pairs of conjugate points of an involution and through a common external point P, they will all meet at a second fixed point Q, such that PQ passes through the centre of the involution. The *Foci* of the involution are the points of contact of the two circles which can be drawn through P and Q so as to touch the transversal: these points are *imaginary* when P and Q lie on opposite sides of the transversal.

110. *A pencil of rays in involution has in general two conjugate rays only at right angles; but if two pairs of conjugate rays be at right angles every two conjugate rays are at right angles.*

Let P be the vertex of a pencil in involution, and let a transversal meet two pairs of conjugate rays in A, A' and B, B'. Then the circles APA' and BPB' determine by their intersection a second fixed point Q, and every circle through P and Q meets the transversal in points C and C' such that PC, PC' are conjugate rays. [Art. 109.

Now in order that the angle CPC' may be a right angle the centre of the circle CPC' must lie on the axis of involution. In general one circle only can be drawn through P and Q so as to have its centre upon a given transversal; but if two circles can be so drawn the transversal must bisect PQ at right angles, and must therefore be the locus of the centres of all the circles that can be drawn through P and Q.

It is hence evident that a right angle turning about its vertex generates a pencil of rays in *negative* involution: for example, *the conjugate diameters of a circle constitute a negative involution.*

It is further evident that there are two points (one on each side of the axis) at which all the segments of a range in negative involution subtend right angles.

111. *If a row of points be in involution any four of them and their four conjugates are equicross, and conversely.*

(i) For if (A, A'), (B, B'), (C, C'), (D, D') be conjugate points in an involution of which O is the centre, then since by definition,

$$OA' : OB' = OB : OA,$$

therefore $$A'B' : OB' = AB : OA;$$

and similarly $$C'D' : OD' = CD : OC.$$

Hence $$A'B'.C'D' : OB'.OD' = AB.CD : OA.OC;$$

and in like manner (or by merely interchanging B and D, and B' and D'),

$$A'D'.C'B' : OB'.OD' = AD.CB : OA.OC;$$

and therefore $\{A'B'C'D'\}$ is equal to $\{ABCD\}$, or every four points and their conjugates are equicross, as was to be proved.

From the general relation $\{ABCD\} = \{A'B'C'D'\}$, where the several couples may be taken arbitrarily, we deduce as one form of the relation between three couples in involution (by substituting A' for D and A for D'), that

$$\{ABCA'\} = \{A'B'C'A\},$$

which readily reduces to

$$AB.B'C'.CA' = A'B'.BC.C'A.$$

Another form of the relation between the three couples is

$$\{ABA'C\} = \{A'B'AC'\},$$

or $$\frac{AB.A'C}{AC.A'B} = \frac{A'B'.AC'}{A'C'.AB'},$$

or $$AB.AB' : AC.AC' = A'B.A'B' : A'C.A'C'.$$

Conversely, when one of these relations is established the three couples AA', BB', CC' are proved to be in involution.

(ii) Otherwise thus. Let circles be drawn each through two conjugate points of an involution so as to cointersect at points P and Q (Art. 109). Join P to any four of the points and join Q to their conjugates. Then it may be easily shewn that the two pencils thus formed have their angles equal (or supplementary) each to each,* and are therefore equianharmonic.

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* For this proof see McDowell's *Exercises on Euclid and in Modern Geometry*, Arts. 206, 7 (Cambridge, 1878).

112. *The foci and any two conjugate points of an involution constitute a harmonic range.*

This is evident from Art. 35, Cor. 1, where A and A' may be taken as the foci of an involution of which S and X are any two conjugate points.

Conversely, any two points S and X which form a harmonic range with a pair of fixed points A and A' are conjugate points in an involution whereof A and A' are the foci.

SCHOLIUM B.

The theory of Involution was laid down by DESARGUES (1593—1662), the friend of Descartes and teacher of Pascal, in his BROUILLON PROIECT *d'une atteinte aux éuénemens des rencontres d'un Cone auec un Plan*, discovered in manuscript by M. Chasles in 1845, and printed in Poudra's two volume edition of the *Œuvres de Desargues*, vol. I. pp. 97—230 (Paris, 1864); an analysis of the work being also given, in which its strange and embarrassing terminology is replaced by the expressions now in use.

The germ of the theory is contained in lib. VII. prop. 130 of the *Collectio* of PAPPUS (p. 873, ed. Hultsch), which may be stated conversely as follows:

If the sides of a triangle PQR meet a transversal in A, B, C, and if the three radiants from any point O to its opposite vertices meet the transversal in F, E, D respectively, then

$$AF.CB : AB.CF = FA.DE : FE.DA.$$

That is to say, the opposite sides and the two diagonals of any quadrilateral $OPQR$ meet any transversal in pairs of points (A, F), (B, E), (C, D) in involution, the cross ratio $\{AFCB\}$ being equal to $\{FADE\}$.

DESARGUES, having defined and established some properties of his *Involution de six Points*, and having enunciated the so-called theorem of Ptolemy in a new form, next shews that the pencil subtended at any vertex by six points in involution is cut in points in involution by any transversal (Poudra, pp. 247, 256—8); he fully establishes the theory of poles and polars (sometimes wrongly attributed to de La Hire), shewing *inter alia* that the three pairs of chords joining four points on a conic determine a self-conjugate triangle, and not omitting to notice also the case of polar planes (pp, 263, 72, 90); and he proves that any conic and the sides of an inscribed quadrilateral determine points in involution on any transversal (p. 268). It is to be observed that he regards six points as constituting a complete involution (which however does not really detract from the generality of his conclusions), and that he uses the term *Involution de quatre Points* to denote the two double points or foci together with a single pair of conjugate points; that is to say, he regards a harmonic range as an involution of four points.

It may be well to quote from Poudra's vocabulary of terms used by Desargues (*Œuvres*, I. 101) the explanation of the still surviving term *Involution*, which signifies "Trois couples de points tels que $oa \times oa' = ob \times ob' = oc \times oc'$ et que tous les points conjugués sont tous mêlés ou démêlés entr' eux."

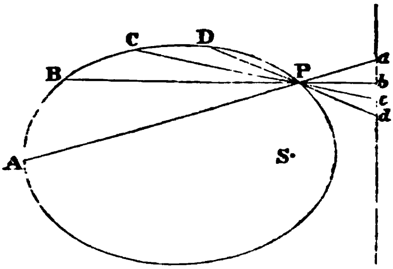
ANHARMONIC PROPERTIES OF CONICS.

PROPOSITION I.

113. *The cross ratios of four fixed points on a conic and of the tangents thereat are constant and equal to one another.*[*]

(i) From four fixed points A, B, C, D on a conic draw chords to any point P on the curve, and produce them to meet the S-directrix in $abcd$ respectively, so that

$$P\{ABCD\} = P\{abcd\} = S\{abcd\}.$$



Then since the angles aSb, bSc, cSd, being equal or supplementary to the halves of the constant angles ASB, BSC, CSD respectively (Art. 13), are themselves constant, it follows that $S\{abcd\}$ and $P\{ABCD\}$ are constant, whatever be the position of P on the curve. Conversely, a conic can in general be drawn through six points† $PP'ABCD$ so related that $P\{ABCD\} = P'\{ABCD\}$; or if $ABCD$ be fixed points, and P a variable point such that $P\{ABCD\}$ is constant, the locus of P is a conic through $ABCD$.

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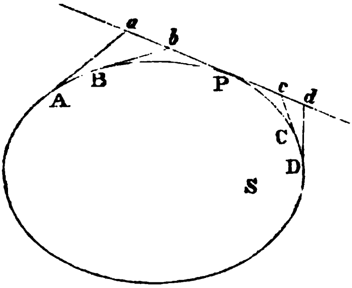
[*] The reciprocal properties (i) and (ii) were stated in their direct and converse forms in STEINER'S *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, pp. 156, 7 (Berlin, 1832), where it is remarked that: "Die Sätze links [(i), (ii)] sind, unter anderer Form abgefasst, bekannt." The property (iii) was given (for the circle) in CHASLES' *Géométrie Supérieure*, § 663 (Paris, 1852).

† The six points in Art. 105 may be regarded as lying on a conic which has degenerated into a pair of straight lines.

(ii) Next let the tangents to a conic at four fixed points $ABCD$ meet the tangent at any fifth point in the range *abcd*.



Then since the angles aSb, bSc, cSd, being equal or supplementary to the halves of the constant angles ASB, BSC, CSD respectively (Art. 13, Cor.), are themselves constant, it follows that $S\{abcd\}$ or $\{abcd\}$ is constant. Conversely, a conic can in general be described touching six straight lines so related that four of them determine ranges of equal cross ratios $\{abcd\}$ and $\{a'b'c'd'\}$ on the remaining two; or if four fixed straight lines meet a variable straight line in a range of constant cross ratio, the variable line envelopes a conic touching the four fixed lines.

(iii) Lastly let P be a fixed point on a conic, PQ a variable chord meeting the S-directrix in R, and let T be the point of concourse of the tangents at P and Q.

Then since the angle RST is always a right angle (Art. 9, Cor. 1),

$$P\{Q\} = P\{R\} = S\{R\} = S\{T\} = \{T\},$$

or the pencil subtended at P by any four positions of Q on the curve is equal to the range in which the tangents at the Q's intersect the tangent at P.

Corollary 1.

If two angles MBD, MCD of given magnitudes turn about B, C as poles in such a manner that the intersection M of two *of their arms describes a straight line, the intersection D of their remaining arms describes (in general) a conic passing through B and C;*[*] for since

$$B\{D\} = B\{M\} = C\{M\} = C\{D\},$$

a conic can be drawn through B, C and any four positions of D; and three assumed positions of D together with B and C determine a single conic on which every other position of D must lie. But if the straight line described by M passes through B or C, or if the pencils $B\{D\}$ and $C\{D\}$ have a common ray, the locus of D is a straight line. The general case affords a ready means of drawing a conic through five given points. [Ex. 367.

Corollary 2.

If three straight lines md, dr, rm turn about given poles B, C, D, whilst m and r move along fixed straight lines PG and PQ, the point d describes a conic passing through B and C; for it is evident that

$$B\{d\} = \{m\} = \{r\} = C\{d\}.$$

It appears (by taking special cases) that the point P and the intersections of BD, PQ and CD, PG likewise belong to the locus: and conversely, if these three points and B and C be given, the lines PG and PQ can be drawn, and the locus of d, which is the conic through the five given points, can be traced.[†]

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[*] This is NEWTON'S method of generating conic sections. The theorem is proved in the *Principia*, Lib. I. Sect. V. lemma 21, where it is deduced from lemma 20 [= Ex. 80], and this again from the theorem *Ad quatuor lineas* (Scholium C). It is also stated, with generalisations and limitations, in his *Enumeratio Linearum Tertii Ordinis*, Cap. VII. (see p. 26 in Talbot's translation, London 1861; or NEWTONI *Opera quæ exstant omnia*, vol. I. 556, ed. Horsley, London 1779), under the head: *De Curvarum Descriptione Organica.*

[†] This method of drawing a conic through five given points was discovered by MACLAURIN in 1722 although not published by him till 1785. In the *Philosophical Transactions of the Royal Society of London* (vol. VIII. p. 50, 1809) he shews how to deduce it by elementary geometry from the above mentioned lemma 20 of the *Principia*, "which itself is a case of this description" (p. 43). Braikenridge, who had already published Maclaurin's construction, is said to have been in communication with him and to have been made acquainted with his theorems in 1727 (*ibid.* pp. 6, 43).

PROPOSITION II.

114. *The diameters of a conic form a pencil in involution in which conjugate diameters are conjugate rays.*

(i) For since any two conjugate diameters meet the S-directrix in points V and V' such that

$$XV.XV' = CX.SX = \text{a constant}, \quad \text{[Art. 14, Cor. 4.}$$

therefore CV and CV' are conjugate rays in a pencil in involution. This involution is *negative* in the ellipse, and *positive* in the hyperbola: in the latter case the asymptotes are the double rays. [Art. 53.

(ii) The centre of the involution determined by the pencil of diameters on any tangent is at its point of contact, since this is evidently conjugate to the point at infinity upon the tangent. This is in accordance with Art. 47, from which a second proof of the proposition may be derived.

Corollary 1.

From a fixed point O draw the perpendicular OP to a variable diameter CP, and produce it to meet the conjugate diameter in D: then since

$$C\{D\} = C\{P\} = O\{D\},$$

the locus of D is a conic through O and C, and it evidently has real points at infinity on the axes of the original conic. At the four points of concourse of the conics the positions of OD are normal at D to the original conic. Hence we infer that *there are four points on a given conic such that the normals thereat cointersect at a given point O, and the four points lie on an equilateral hyperbola, which passes through O and through the centre of the given conic, and has its asymptotes parallel to the axes of the latter.*[*]

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[*] This method of drawing a normal OD to a conic from a given point O, with the help of an equilateral hyperbola, is given by APOLLONIUS in his *Conics*, Lib. V. props. 58–63, where he regards the normal as a line drawn from O so that the intercept upon it between D and the axis of the given conic is a minimum. For another treatment of normals see Prof. CREMONA'S article *On Normals to Conics*, in the *Oxf. Camb. and Dublin Messenger of Mathematics*, vol. III. p. 88. See also Ex. 286,

Corollary 2.

By drawing OP to meet CP (Cor. 1) at any other *given angle* and in a given "sense" or direction of rotation, and proceeding as above, we determine a conic passing through O and C and having its asymptotes (real or imaginary) parallel to those conjugate diameters of the given conic which contain an angle equal to the given angle; and the two conics intersect at four points D such that OD meets the given conic at the given angle. If the sense of rotation be not given, four other positions of OD (making in all eight), which meet the given conic at the given angle, can be in like manner determined.

SCHOLIUM C.

The celebrated problem of the "*Locus ad quatuor lineas*" (τόπος ἐπὶ δ' γραμμάς)—handed down by PAPPUS from his predecessors without solution (*Collectio* Lib. VII, vol. II. pp. 677—9, ed. Hultsch), solved only by his new method of coordinates by DESCARTES (*Geometria* Libb. I. II. 7—16, 24—34, ed. Schooten, 1659), and at length completely solved by NEWTON (*Principia* Lib. I. Sect. V. lemm. 17—19) by the elementary geometry of Apollonius—implicitly contains the fundamental anharmonic property of four points on a conic (Art. 113). The problem and its Newtonian solution are as follows.

(i) *If P be any point of a conic and $ABDC$ a given inscribed trapezium, and if straight lines PQ, PR, PS, PT meet the sides AB, CD, AC, DB respectively at given angles: the rectangle $PQ \times PR$ is to the rectangle $PS \times PT$ in a given ratio.*

a. First let PQ and PR be parallel to AC, and PS and PT parallel to AB; and let the side BD be also parallel to AC. Then since $PQ.QK$ varies as $AQ.QB$, "per prop. 17, 19, 21 & 23 lib. III. conicorum *Apollonii*" (Art. 16, Cor. 1), if K be the point in which PQ meets the conic again; and since the diameter of the chords AC, BD, KP bisects also the intercept QR, so that $QK = PR$; it follows that $PQ.PR$ varies as $AQ.QB$ or $PS.PT$, as was to be proved.

b. Next let BD be not parallel to AC.[*] Draw Bd parallel to AC meeting ST in T and the conic in d. Join Cd cutting PQ in r,

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note, where if T and U be given T' and U' can be at once determined; and thus from the intersection of any two normals to a conic two other normals can be drawn. Corollaries 1 and 2, as they stand, are taken from CHASLES' *Traité des Sections Coniques*, chap. VII. pp. 142–4. On the parabola see Ex. 617, note.

[*] See the lithographed figure No. 3.

and draw DM parallel to PQ cutting Cd in M and AB in N. Then by similar triangles, and by parallels, Bt (or PQ) is to Tt as DN to NB; and Rr is to AQ (or PS) as DM to AN; and therefore

$$\begin{aligned} PQ.Rr : PS.Tt &= \text{rect. } NDM : \text{rect. } ANB,^{*} \\ &= PQ.Pr : PS.Pt, \text{ by case } a, \\ &= PQ.PR : PS.PT, \text{ dividendo.} \end{aligned}$$

c. Having thus shewn that this last ratio has the constant value $DN.DM : AN.NB$, we see at once that $PQ.PR$ will still vary as $PS.PT$ if PQ, PR, PS, PT be drawn each at its own constant inclination to AB, CD, AC, DB respectively. It is further evident that if PX and PY be drawn to meet the diagonals AD and BC at constant angles, each of the ratios $PX.PY : PQ.PR : PS.PT$ is constant. Conversely (lemma 18), if $PQ.PR$ varies as $PS.PT$, the locus of P is a conic circumscribing $ABCD$. In lemma 19 and its two corollaries NEWTON completes the solution of the problem by shewing how to determine the actual locus of P for a given value of the ratio $PQ.PR : PS.PT$; and he concludes by remarking with evident satisfaction: "Atque ita problematis veterum de quatuor lineis ab *Euclide* incœpti & ab *Apollonio* continuati non calculus, sed compositio geometrica, qualem veteres quærebant, in hoc corollario exhibetur."

(ii) The anharmonic property of four points on a conic follows immediately from the above theorem *Ad quatuor lineas*. For if PQ, PR, PS, PT be perpendiculars to AB, CD, AC, BD respectively (cf. Art. 94), so that $PQ.AB = PA.PB \sin APB$, &c., and thus

$$\frac{PQ.PR.AB.CD}{PS.PT.AC.BD} = \frac{\sin APB}{\sin APC} \cdot \frac{\sin CPD}{\sin BPD} = P\{ABCD\};$$

and if the ratio $PQ.PR : PS\,PT$ and the trapezium $ABDC$ be given; the cross ratio $P\{ABCD\}$ will be constant.

(iii) CHASLES and others have proved the constancy of the cross ratios of four given points on or tangents to a conic by projection from the circle, and have taken the properties thus proved as the foundation of their treatises on conics: but the most elementary proofs of the properties in question are those which we have adopted in Art. 113 from the *Geometrical Demonstration of the Anharmonic Properties of a Conic* contributed by Mr. B. W. HORNE, Fellow of St. John's College, to the *Quarterly Journal of Mathematics*, vol. IV. 278 (1861): his proofs assume only those simple angle-properties of the focus of a conic which reduce, when the directrix is removed to infinity, to the fundamental angle-properties of the circle (Scholium A, p. 22).

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* This notation was formerly in use for the rectangle $AN.NB$.

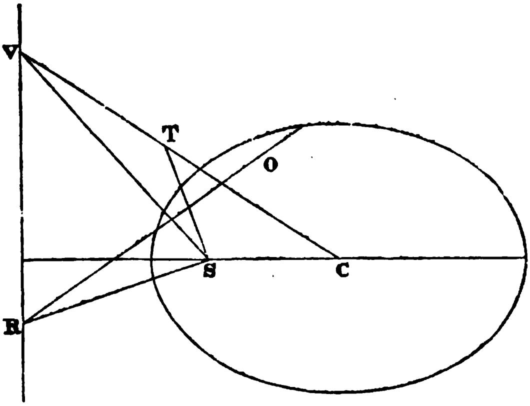
RECIPROCAL POLARS.

PROPOSITION III.

115. *The polar of any point with respect to a conic is a straight line, and conversely.*[*]

(i) Let a variable chord be drawn to a conic through a given point O, and let the tangents at its extremities intersect in T; then will the locus of T be a straight line.

For if the variable chord and its diameter CT meet the directrix in R and V respectively, then since SR, ST and OR, SV are at right angles (Arts. 9, 14), therefore



$$S\{T\} = S\{R\} = O\{R\} = S\{V\} = C\{T\}.$$

And since the homographic pencils $S\{T\}$ and $C\{T\}$ have a common ray CS (the point T lying on the axis when OR is a principal ordinate), the locus of T is a straight line. [Art. 105.

Conversely, if T be any point on a straight line, it may be shewn in like manner that its chord of contact determines homographic ranges $\{R\}$ and $\{\infty\}$ on the directrix and the straight line at infinity, and hence that it passes through a fixed point. [Art. 104.

(ii) Otherwise thus. Let PL and PN be given tangents to a conic, PMR any chord through P, and O its intersection with LN; and let A and B be the intersections of the tangents at N, M and N, R.

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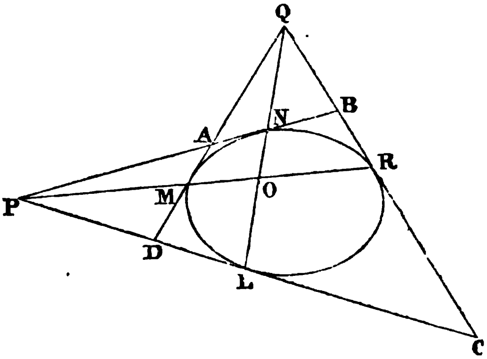
[*] For other proofs of the properties of polars see Arts. 17, 18, 41, 93.

Then since the cross ratios of the tangents at L, M, N, R



and of their points of contact are equal (Art. 113), therefore (estimating the former on the tangent at N and the latter on the transversal MR) we have

$$\{PANB\} = L\{LMNR\} = \{PMOR\};$$

whence it follows that the tangents AM and BR intersect on the fixed straight line NO, and conversely. [Art. 104.

PROPOSITION IV.

116. *A row of points on any axis and their polars with respect to a conic are homographic, and they determine an involution on that axis.*

(i) It follows at once from the construction of Art. 115 (i), where RST is a right angle, that if T be taken on the polar of a given point O, then

$$\{T\} = S\{R\} = O\{R\},$$

or the points T and their polars OR are homographic.

(ii) Otherwise thus. In the preceding figure, if the equal cross ratios $L\{LMNR\}$ and $N\{LMNR\}$ be estimated on the transversal MR, then

$$\{PMOR\} = \{OMPR\},$$

or O and P are harmonic conjugates with respect to M and R. [Art. 106.

Hence, as P moves along MR, the points O and P are couples in the involution of which M and R are the foci (Art. 112); that is to say, if P be a variable point on the polar of any given point Q, and QO be the polar of P, the range $\{P\}$ is homographic with the pencil $Q\{O\}$, and the two together determine an involution on the polar of Q.

Corollary 1.

If a pair of points divide a chord of a conic harmonically, each point lies on the polar of the other, and the two are said to be *conjugate with respect to the conic.* From the reciprocity of the relation between such points it is easy to deduce the theorem of Art. 17, Cor. 1, that *the intersection of any two straight lines is the pole of the line which joins their poles.* It is evident that *a system of conics having a common self-polar triangle* (note, p. 272) *determine an involution on any transversal drawn through a vertex of the triangle.*

Corollary 2.

Two straight lines are said to be *conjugate with respect to a conic when they pass each through the polar of the other*, or in other words, when they are harmonic conjugates with respect to the two tangents (real or imaginary) that can be drawn to the conic from their point of concourse. From a given point there can in general be drawn one pair only of straight lines *at right angles* to one another and conjugate to a given conic (Art. 110); but *if the given point be a focus, every two conjugate lines drawn through it are at right angles* (Art. 7); and conversely it may be shewn that no other real point is so related to the conic. Notice that conjugate diameters are also conjugate lines in the sense of this corollary.

PROPOSITION V.

117. *If the locus of a point be a conic the envelope of its polar with respect to a conic is a conic, and conversely.*

Take four fixed points $ABCD$ and their polars with respect to a conic: and take a variable point P and its polar with respect to the same conic: and let the fixed polars meet the polar of P in the points a, b, c, d respectively.

Then by Art. 116, the points a, b, c, d are the poles of PA, PB, PC, PD; and $P\{ABCD\}$ is equal to $\{abcd\}$; and if the one be constant the other is constant.

Hence, if the locus of P be a conic passing through the points $ABCD$, the envelope of the polar of P will be a conic touching the polars of $ABCD$, and conversely. [Prop. I.

Corollary.

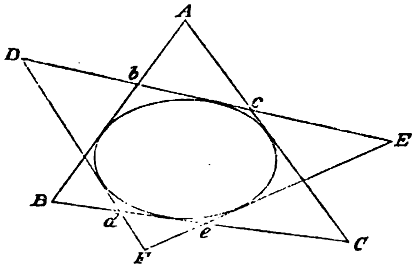
To any figure generated by points or *poles* corresponds a reciprocal figure generated by their *polars* with respect to any assumed auxiliary conic; and any property of the one figure implies a reciprocal property of the other. The method of deducing reciprocal properties from one another will form the subject of Chapter XII. Notice, as a special case of the proposition, that any conic may be regarded as its own reciprocal, its several points being the poles of the tangents thereat. Also see Scholium E at the end of this chapter.

THE TRIANGLE.

PROPOSITION VI.

118. *If two triangles circumscribe a conic their six vertices lie on a conic, and conversely.*

Let ABC and DEF be two triangles whose six sides touch the same conic: let BC meet DF in d and EF in e: and let DE meet AB in b and AC in c.



Then by Prop. I, since the four tangents AB, AC, FD, FE are homographic with respect to the tangents BC and ED, therefore

$$\{BCde\} = \{bcDE\},$$

or

$$F\{BCDE\} = A\{BCDE\};$$

and therefore the six points $ABCDEF$ lie on one conic.

Conversely, if the six vertices of two triangles lie on a conic, it may be shewn in like manner that their six sides touch a conic.

Corollary 1.

If the enveloped conic be a parabola, and if D be taken at its focus and E and F at the circular points at infinity (Chap. XI), the conic through $ABCDEF$ becomes a circle. Hence *the circumscribed circle of any triangle ABC whose sides touch a parabola passes through the focus* (Art. 29).

Corollary 2.

If two conics be so related that a single triangle can be inscribed in the one and circumscribed to the other, an infinity of triangles can be so circuminscribed to the two conics. For let ABC be the first triangle, and ab any chord of the outer conic which touches the inner: complete the triangle abc by drawing the second tangents from a and b to the inner conic: then the point c must always lie on the same fixed conic with the points $ABCab$.

PROPOSITION VII.

119. *If two triangles be self-polar* with respect to a conic, their six vertices lie on a conic, and their six sides touch a conic.*

Let ABC and DEF be two triangles self-polar with respect to a conic:† then evidently the join of any two of their six sides is the pole of the join of the opposite vertices. [Art. 116.

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* A triangle may be called *self-polar* with respect to a conic— cf. the French term "*autopolaire*"—when each vertex is the pole of the opposite side. Such triangles are usually termed *self-conjugate*, and some writers call them *self-reciprocal.* The vertices of a self-conjugate triangle constitute what is called a *Conjugate* (or self-conjugate) *Triad* of points.

† See the lithographed figure No. 4.

Let BC meet AD in d and EF in d': then d' is the pole of AD, and d and d' are therefore conjugate points (Art. 116) with respect to the conic.

Also let BC meet AE in e and DF in e': then e and e' are likewise conjugate points with respect to the conic: and it is evident that B and C are conjugate with respect to it.

It follows that the points BC, dd', ee' are couples in involution, and hence (Arts. 111, 102) that

$$\{BCde\} = \{CBd'e'\} = \{BCe'd'\},$$

or

$$A\{BCDE\} = F\{BCDE\}.$$

Therefore a conic goes through the six points $ABCDEF$; and therefore another conic (Prop. VI) touches the six sides of the two triangles.

Corollary 1.

If upon a given conic one triad of points self-conjugate with respect to a second given conic can be determined, an infinity of such triads can be determined upon it. For if ABC be the first triad, R any other point on the first conic, S one of the points in which the polar of R with respect to the second conic meets the first, and T the pole of RS with respect to the second conic; the point T must always lie on the same conic with $ABCRS$. By taking R at the centre of an equilateral hyperbola, and S and T at the circular points at infinity (Chap. XI.), we deduce that *the circle through any conjugate triad with respect to an equilateral hyperbola passes through its centre.* [Art. 64, Cor. 4.

Corollary 2.

Let the first conic become a circle, and let Q be one of its points of concourse with the second conic. Next let the points R and S coalesce at Q, so that the inscribed self-polar triangle RST degenerates into the vanishing triangle QQT: then T becomes *the pole with respect to the conic of the tangent to the circle at Q.* Conversely, if QQ' be any chord of a conic and T its pole, the circle drawn through T so as to touch QQ' at Q (or Q') is such that an infinity of triangles self-polar with respect to the conic can be inscribed in it.

Corollary 3.*

Let the circle described as above meet the diameter CT of the conic in a second point t, and let V be the middle point of QQ', and CP, CD the semi-diameters conjugate and parallel to QQ'. Then since

$$VT.Vt = QV^2 = \frac{CD^2}{CP^2}(CP^2 \sim CV^2) = CD^2.\frac{CT \sim CV}{CT}$$

it follows that $CT.Vt$ is equal to CD^2, and hence that

$$CT.Ct = CP^2 \pm CD^2 = CA^2 \pm CB^2;$$

and hence that *the circumscribed circle of any triangle self-polar with respect to a conic is orthogonal to its director circle.* In the case of the parabola every such circle is orthogonal to the directrix, or *has its centre upon the directrix.* Otherwise thus: let any circle in which self-polar triangles with respect to a conic can be inscribed meet the director circle in O and the chord of contact of the tangents OQ, OQ' to the conic in P and P'; and let V be the middle point of QQ'. Then since the points P and P' are conjugate, and since the angle QOQ' is a right angle,

$$VP.VP' = VQ^2 = VO^2;$$

and therefore the line VO, which is normal to the director circle at O, is the tangent at O to the circle OPP'.

THE QUADRILATERAL.

PROPOSITION VIII.

120. *The intersections of the opposite sides and of the diagonals of a quadrilateral are a conjugate triad with respect to every conic circumscribing the quadrilateral.*

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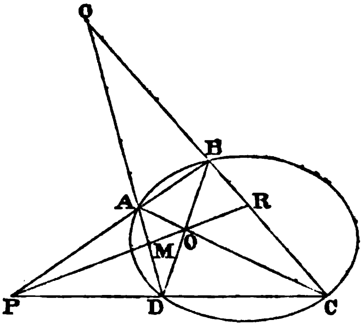
* The former of the two proofs of GASKIN'S theorem (note, p. 186) in Cor. 3 is due to M. Paul Serret (*Nouvelles Annales* XX. 79, 1861). The theorem may also be proved independently of Prop. VII., as follows: let QQ' be any chord of an ellipse, V and T its middle point and pole, and P, R any two conjugate points upon it, so that $VP.VR = QV^2$. Let the circle round TPR meet VT again in O: then it may be shewn that $VO.VT = QV^2$, and hence that $CO.CT = CA^2 + CB^2$ (*Nouv. Ann.* XX. 25). See also the *Quarterly Journal of Mathematics* X. 129.

If $ABCD$ be any four points on a conic, and OPQ the intersections of (AC, BD), (AB, CD), (AD, BC); then since



the line OP and the point Q divide both AD and BC harmonically (Art. 107), therefore OP is the polar of Q.

In like manner OQ is the polar of P: and therefore O is the pole of PQ, and the points OPQ are a conjugate triad, as was to be proved.

Corollary 1.

To draw tangents to a conic from a given point Q *with the ruler only*, draw any two chords QAD and QBC from the given point: let the line PO (the join of the joins of AB, CD and AC, BD) cut the conic in T and T': draw QT and QT', which will be the tangents required.

Corollary 2.

From a given point P draw a fixed chord PAB and a variable chord PDC to a conic. Then since AC, BD and AD, BC meet on the polar of P, therefore

$$A\{C\} = B\{D\} = A\{D\}, \qquad \text{[Prop. I.}$$

where C (or D) may be *either* extremity of the variable chord. Hence, taking any three positions of CD, we have

$$A\{CC'C''D\} = A\{DD'D''C\};$$

and therefore *any variable chord CD drawn through a fixed* *point determines an involution at any point A on the conic,*[*] *and conversely.* For example, a chord of a conic which subtends a right angle (Art. 110) at a given point on the curve passes through a fixed point on the normal thereat.[†]

PROPOSITION IX.

121. *The three diagonals of a complete quadrilateral determine a triangle which is self-polar with respect to every conic inscribed in the quadrilateral.*

Let a, b, c, d denote the tangents at any four points A, B, C, D on a conic, and ab the join of any two of them a and b.

Then (in the preceding figure), since AB and CD pass through P, their poles ab and cd lie on OQ the polar of P. And in like manner ad and bc lie on OP, and ac and bd on PQ.[‡]

That is to say, the three diagonals of the circumscribed quadrilateral $abcd$ lie upon the sides of the self-polar triangle OPQ.

Corollary.

In the reciprocal quadrilaterals $abcd$ and $ABCD$ determined by any four tangents to a conic and their points of contact respectively, two pairs of diagonals cointersect and form *a harmonic pencil* $O\{PAQB\}$ (Art. 107); and the third diagonals lie in one straight line, and their extremities form *a harmonic range* $\{ac, P, bd, Q\}$. [Prop. IV.

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[*] Otherwise thus: if PE and PF be the tangents to the conic from the given point P, then $E\{ECFD\}$ is harmonic (Art. 18), and therefore $A\{ECFD\}$ is harmonic, or AC and AD are conjugate rays in the involution of which AE and AF are the double rays. Note that four *points on a conic* are said to be *harmonic* when they subtend a harmonic pencil at every fifth (Prop. I.); and the tangents at four such points are said to be harmonic.

[†] This theorem is due to Frégier (Gergonne's *Annales* VI. 231, 1816).

[‡] This was proved by MACLAURIN in Sect. II. §§ 35, 36 of the *Appendix* to his work on algebra above referred to (Ex. 324, note), in which he applied COTES' theorem of harmonic means to curves of the second order. He thus virtually reciprocated a theorem of Desargues (Prop. VIII.), although reciprocation, as a method, was only discovered in the century following.

PROPOSITION X.

122. *Any conic and the three pairs of chords joining any four points upon it meet every transversal in four pairs of points in involution.*[*]

(i) For if any transversal meet a conic in the points AA', and any two of the three pairs of chords joining four points on the curve in the points BB' and CC', then by the theorem *Ad quatuor lineas* (Scholium C),

$$AB.AB' : AC.AC' = A'B.A'B' : A'C.A'C';$$

and therefore AA', BB', CC' are couples in an involution (Art. 111); and the third pair of connecting chords determine a fourth couple in the same involution.

(ii) Otherwise thus. Let any transversal meet a conic in the points AA', and any two chords ab and cd in BB', and the chords ad and bc in CC'.

Then since the points $AdbA'$ on the conic are equicross with respect to a and c, therefore (estimating $a\{AdbA'\}$ and $c\{AdbA'\}$ on the transversal) we have

$$\{ACBA'\} = \{AB'C'A'\} = \{A'C'B'A\};$$

and therefore AA', BB', CC' are couples in an involution

(iii) The four sides of a given trapezium suffice to determine an involution on any transversal (Art. 108), and every conic circumscribing the trapezium passes through an additional couple in such involution; and no conic which does not circumscribe the trapezium can pass through a couple in such involution for every position of the transversal. For if the transversal, as it turns about any point A of a conic, meets it again always in the conjugate point A', this conic must have an infinity of points A' in common with the conic through A and the four summits of the trapezium. Hence the proposition may be stated as follows:

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[*] This is one of the fundamental theorems of DESARGUES. Having first proved it for the circle he extended it to the general conic by projection, leaving to others to devise some direct proof applicable to the general case (Poudra's *Œuvres de Desargues* I. 174, 198). The proof given above in Art. 122 (i) shews that the theorem is an immediate corollary from the ancient theorem *Ad quatuor lineas*. For the second proof (with a diagram) see Salmon's *Conic Sections*, Art. 344. The theorem seems to have been first stated for the case of *three conics*, instead of one conic and an inscribed quadrilateral, by Sturm (Gergonne's *Annales* XVII. 180). At the end of the same memoir (p. 198) Sturm alludes to the reciprocal theorem of Prop. XI. See also Poncelet *Traité des Propriétés Projectives* II. 149.

A system of conics through four common points (with their three pairs of common chords) meet every transversal in pairs of points in involution, and conversely.*

Each of the two double points belonging to any transversal must be at its point of contact with one of the conics of the system, or at an intersection of a pair of their common chords. There can therefore in general be drawn *two conics* through four given points to touch a given straight line.

Corollary 1.

The foci of any transversal† with respect to a quadrilateral are evidently *conjugate points* with respect to every conic circumscribing it. Hence *the polar of a given point F with respect to a system of conics through four given points $ABCD$ passes through a fixed point F'*,‡ which is determined as the second focus of the transversal which touches the conic $ABCDF$ at F.

Corollary 2.

Through three given points ABC draw two conics touching a given line at its extremities F and F' respectively. These conics intersect at a fourth point D, through which must pass every conic through ABC which has F, F' for a pair of conjugate points.§ Hence we are led to infer generally that, in describing conics subject to given conditions, *the condition that two specified points F and F' should be conjugate with respect to a conic is equivalent to having given one point D on the curve.*

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* Each pair may be regarded as a degenerate hyperbola of the system.

† We use this expression as an abbreviation for "the foci of the involution determined upon any transversal by the sides of the quadrilateral, taken in opposite pairs."

‡ It was proved analytically by Lamé (*Examen des différentes méthodes employées pour résoudre les problèmes de Géométrie* pp. 34—38, Paris, 1818) that *if a system of conics (or quadrics) have the same points of intersection, their diameters (or diametral planes) severally conjugate to a system of parallel diameters meet in a point.*

§ For example, if F and F' be the circular points at infinity, the conics are equilateral hyperbolas, and they pass through the orthocentre of the triangle ABC.

Corollary 3.

Each common tangent FF' to two conics is cut harmonically by every other conic (or pair of chords) through their four common points. By supposing three of the four common points to coalesce, and one of the conics to become a circle, we deduce that the common chord of a conic and its *circle of curvature* at any point and their common tangent thereat divide their further common tangent harmonically. [Ex. 647.

Corollary 4.

A system of conics having *double contact* cut every transversal in pairs of points in an involution, having one focus upon their common chord of contact CC'. In particular, if a transversal meet one of the conics in AA', and their fixed common tangents (which themselves constitute a degenerate conic of the system) in BB', then CC' passes through a focus of the involution AA', BB'.* Hence, if two points AA' on a conic and also two tangents to it be given, their chord of contact passes through one or other of two fixed points on the line AA'; and if a third point A'' on the conic be given, the same chord of contact passes also through one of two fixed points on the line AA'', and may therefore have any one of *four* positions. There are therefore four solutions of the problem, *to draw a conic through three given points so as to touch two given lines.*

Corollary 5.

If one focus of an involution be at infinity its other focus bisects every segment of the involution (Art. 112). Hence and from Cor. 4, *any two conics having double contact make equal and opposite intercepts on every transversal parallel to their common chord*, and therefore on *every* transversal without exception in the case in which their common chord is the straight line at infinity (Art. 57). Conversely, we are led to infer from Ex. 50 that *every two similar and coaxal conics are to be regarded as having double contact with one another upon the line at infinity.*

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* Notice the special case of Ex. 69.

PROPOSITION XI.

123. *The pairs of tangents from any point to a system of conics inscribed in the same quadrilateral form a pencil in involution, and conversely.*

Let four tangents to a conic intersect in the three pairs of points aa', bb', cc'; and let any transversal meet their polars in AA', BB', CC' and the conic in DD'; and let O be the point of which the transversal is the polar.

Then each ray of the pencil $O\{aa'bb'cc'DD'\}$ is the polar of the corresponding point in the range $\{AA'BB'CC'DD'\}$, and the two systems are therefore homographic (Prop. IV); and since the latter is in involution (Prop. X) the former is also in involution.

If now we suppose the four tangents to remain fixed whilst the conic varies, the pairs of tangents OD, OD' from any assumed point O to every conic of the system are conjugate rays in the involution $O\{aa'bb'cc'\}$, as was to be proved.

Conversely it may be shewn (Art. 122 § iii) that if the tangents from every point O to a conic belong to the involution $O\{aa'bb'cc'\}$, the conic must be one of the system inscribed in the quadrilateral whose summits are aa', bb', cc'.

Corollary 1.

*The director circles of all conics inscribed in the same quadrilateral are coaxal.** For if O be taken at either point of concourse of any two of these circles, the tangents from it to their two conics will be at right angles, and therefore the tangents from it to every conic of the system will be at right angles (Art. 110). To the same coaxal system belong the circles on the three diagonals of the quadrilateral as diameters. [Art. 33, Cor. 3.

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* This is one of GASKIN'S theorems, for the reciprocal of which see Art. 69. It may also be deduced from Art. 119, Cor. 3. combined with Prop. XII., which require that the *limiting points* of every two of the director circles should lie upon a fixed straight line, and also upon the circle through the intersections of the three diagonals of the quadrilateral, and should therefore be two fixed points. Prof. TOWNSEND has established the analogous theorem in three dimensions, that *the director spheres of the system of quadrics touching eight fixed planes* (*and therefore inscribed in the same developable surface*) *have a common radical plane.* See the *Quarterly Journal of Mathematics* vol. VIII. 10—14. The same theorem appears to have been proved independently by M. PIOQUET (Chasles *Rapport sur les progrès de la Géométrie* p. 370, Paris, 1870). It occurs to me that the director circle and sphere might have been called the ORTHOCYCLE and ORTHOSPHERE.

Corollary 2.

If one side of the quadrilateral be at infinity, its three diagonals become the parallels through the vertices to the opposite sides of a triangle; and the circles upon these diagonals become the perpendiculars of the triangle, whose intersection must therefore be a point on the directrix of every *parabola* inscribed in the triangle. [Art. 29, Cor. 1.

Corollary 3.

It may be shewn by reciprocation[*] from Art. 122 (or directly, by the kind of reasoning there employed), that *the pole of a given straight line with respect to a system of conics inscribed in a quadrilateral lies upon a fixed straight line;* and that to have given that a specified pair of straight lines are conjugate with respect to a conic is equivalent to having given *one tangent* to the curve; and that *two* conics can in general be drawn through one given point so as to touch four given lines; and that *four* conics can be drawn through two given points so as to touch three given lines.

Corollary 4.

Every pair of tangents TP, TQ to a conic whose foci are S and H are harmonic conjugates with respect to the bisectors of the angle STH (Art. 50), as are also the lines from T to the circular points at infinity ϕ and ϕ' (Chap. XI). The tangents TP, TQ are therefore a couple in the same involution with $T\{SH\phi\phi'\}$, and *every conic which has* S *and* H *for foci may accordingly be regarded as inscribed in the trapezium* $S\phi H\phi'$. On account of this affinity of the points ϕ, ϕ' to the foci of every conic in their plane we shall sometimes speak of them as the FOCOIDS, comparing the use of the term *centroid*, or quasi-centre.

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[*] Notice that the proof of Prop. XI. is itself an example of reciprocation.

PROPOSITION XII.

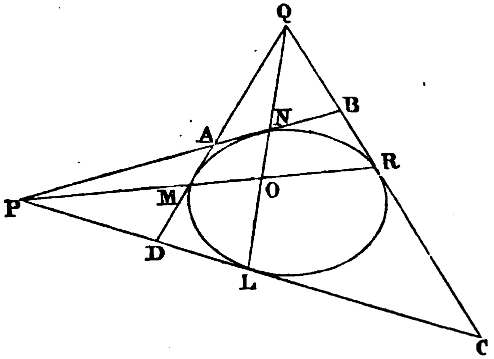
124. *The locus of the centres of all conics inscribed in a given quadrilateral is a straight line,** *which also bisects its three diagonals.*

(i) Since the director circles of the system of inscribed conics are coaxal (Art. 123, Cor. 1), their centres, which are also the centres of their conics, lie on a straight line. This line is evidently the diameter of the quadrilateral (Art. 107), since the middle points of the three diagonals (regarded as flat conics inscribed in the quadrilateral) belong to the locus of centres.

The proposition also follows as a special case from Art. 123, Cor. 3 by regarding the centre of every conic as the *pole of the line at infinity* with respect to it.

(ii) In the following proof the parallelogram of forces is assumed.

Let any conic touch four fixed lines AB, BC, CD, DA in N, R, L, M respectively. Then the resultant of AM and AN, regarded as representing forces, bisects the chord of contact MN, and therefore passes through the centre of the conic.



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* This important theorem of NEWTON (Ex. 505, note), which was originally proved by the elementary method of Exx. 370–2, served as a starting point for later researches into the properties of systems of conics subject to less than five conditions. Notice the use made of it by BRIANCHON and PONCELET in Gergonne's *Annales* XI. 219. It might have been deduced from it by projection—a method not unknown to NEWTON—that there are an infinity of pairs of straight lines conjugate to the entire system of conics touching four given lines, which is the equivalent of Prop. XI. on *involution*.

Thus the centre is a point on the resultant of each of the pairs (AM, AN), (NB, RB), (CR, CL), (LD, MD); that is to say, it is a point on the resultant of AB, CB, AD, CD, and its locus is therefore a straight line.* It is evident that the resultant of these forces bisects the two diagonals AC and BD; and by resolving them severally into

$$PB+AP, \quad CQ+QB, \quad QD+AQ, \quad CP+PD,$$

we see that it bisects the third diagonal PQ also.

Corollary.

One conic and one only can be inscribed in a given quadrilateral so as to have its centre upon any given straight line, since this line by its intersection with the diameter of the quadrilateral determines a single position of the centre of the conic. Hence we are led to infer that *to have given a diameter of a conic is equivalent to having given a tangent to it*, since either datum alike (when four other tangents are given) determines one conic and one only. This is in accordance with Art. 123, Cor. 3, since every diameter of a conic is conjugate to the line at infinity.

PROPOSITION XIII.

125. *The centres of all the conics which circumscribe a given quadrilateral lie upon its eleven-point conic.*

(i) Through four given points *two* conics can be drawn so as to have their centres (real or imaginary) upon any given straight line. [Prop. XII. Cor.

The locus of the centres of all the conics through four given points $ABCD$ is therefore of the second order, since every straight line meets it in two points and two only.

The join AB of any two of the four points meets this conic of centres in two points, which must evidently be the middle point of AB and its intersection with CD.

(ii) Otherwise thus. If O be the centre of any conic through $ABCD$, the radiants from O parallel to the six joins of the points $ABCD$ are homographic with the radiants from O to their middle points (Prop. II); and the locus of O is therefore a conic through those middle points. And it is obvious that the three intersections (AB, CD), (AC, BD), (AD, BC) are also points on the locus.

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* *Nouvelles Annales* I. 24 (1862). For a proof depending upon the dynamical principle of moments see the *Quarterly Journal of Mathematics* VI. 215.

(iii) The conic of centres will have *two* real points at infinity or *one*, or *none* according as two real parabolas or one or none can be circumscribed to the quadilateral $ABCD$. The locus will therefore be in general a hyperbola if this quadrilateral be convex (Ex. 184), or a parabola if two of its sides be parallel, and an ellipse if the quadrilateral be reentrant.

(iv) Let either of the two parabolas (real or imaginary) which pass through $ABCD$ touch the line at infinity in K, which will accordingly be the pole of that line with respect to the parabola, and therefore the *centre* of the parabola.

The conic of centres therefore passes through the two points on the line at infinity which are conjugate with respect to all the conics through $ABCD$ (Art. 122, Cor. 1), as well as through the six middle points and the three intersections of their three pairs of common chords; and we have therefore called it the *Eleven-Point Conic* of the quadrilateral $ABCD$. Its centre *is at the centroid of the points* $ABCD$, since at that point the joins of the middle points of (AB, CD), (AC, BD), (AD, BC) meet and bisect one another.*

It is evident that the *polars of any point on the eleven-point conic of* $ABCD$ *with respect to all the conics round* $ABCD$ *are parallel*, since they all meet in a point (Art. 122, Cor. 1), and one of them is the line at infinity.

Corollary 1.

Since the eleven-point conic E contains a conjugate triad (Art. 120) with respect to every conic F through $ABCD$, therefore if O be an intersection of E and F, the tangent to E at O has its pole with respect to F on E. [Art. 119, Cor. 3.

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* This is at once evident, since four equal particles at $ABCD$ balance two and two about the middle points of any pair of the above lines; and therefore the centroid of the four particles is at the middle point of the line joining any such pair of middle points. In the *Quarterly Journal of Mathematics* VI. 127 I have shewn how to verify a simple construction for the centroid of the *area of any quadrilateral* by an extension of the barycentric principle.

Corollary 2.

The eleven-point conic of any quadrilateral $ABCD$ inscribed in a conic touches the diameter of the quadrilateral formed by the tangents to it at $ABCD$, since the complete locus of centres of the system of conics inscribed in the latter quadrilateral is its diameter (Prop. XII.), and the locus of centres of all conics round $ABCD$ is its eleven-point conic, and *one* conic only can be both inscribed in the one quadrilateral and circumscribed to the other.

Corollary 3.

When the two points at infinity which are conjugate to all the conics round $ABCD$ are the *circular points* the eleven-point conic becomes the nine-point circle, and the points $ABCD$ become a triad and their orthocentre. The nine-point circle really belongs to this form of tetrastigm, and not specially to any one of the four triangles determined by its vertices; in the same way that the system of equilateral hyperbolas circumscribing any one of these four triangles is a system of conics circumscribing the tetrastigm.

SCHOLIUM D.

We have seen that an ellipse or a hyperbola may degenerate into a straight line AA' or its complement (Art. 33, Cor. 3). For example, the diagonals of a quadrilateral may be regarded as flat conics inscribed in it, and accordingly their middle points belong to the locus of centres of all conics inscribed in it (Prop. XII). This agrees with the bifocal definition $SP \pm HP =$ a constant, in accordance with which the point P may in the limit lie anywhere upon the line SH; or upon the complement of SH, if the lower sign be taken.

Again, if TP, TQ and TP', TQ' be the tangents from any point T to two ellipses whose common foci are S and H, the angles PTP' and QTQ' are always equal; and hence when the inner ellipse assumes the line-form SH the angles STP and HTQ are equal. But since this is also the case when the second ellipse is left out of consideration, and the lines TS and TH are simply drawn through the fixed points S and H, the point-pair S and H are so far indistinguishable from the flat conic SH.

Again, let the ellipse be regarded as the envelope of a straight line subject to the relation $\lambda\mu = b^2$, where λ and μ are the perpendiculars upon it from S and H. When b vanishes the ellipse again appears to coincide with the points S and H, which are represented by $\lambda = 0$ and $\mu = 0$ taken *separately;* but by taking $\lambda = 0$ and $\mu = 0$ simultaneously we find that (besides the tangents whose points of contact are at S and H) the limiting conic has an infinity of tangents which ultimately coincide with the line joining S, H and have their points of contact *distributed along* SH.* We may therefore say (1) that an ellipse degenerates into the line SH joining its foci when its minor axis vanishes, meaning that SH is an actual limiting form of the curve;† or (2) we may say that it "degenerates into" the point-pair S, H, understanding that at the instant at which it does so degenerate it ceases to belong properly speaking to the class conic, although the point-pair S, H and the line joining them may be, as regards some properties, indistinguishable.

In like manner the hyperbola may be said to degenerate into its asymptotes ECe and $E'Ce'$ (Art. 54); but strictly speaking it becomes the pair of vertically opposite angles ECE' and eCe', and then has for its tangents at C those lines only through C which lie *within* the said angles. The conjugate hyperbola at the same time becomes coincident with the two supplementary angles, and has for its tangents at C all the lines through C which fall within those angles. It is therefore practically sufficient to say that either of the two conjugate hyperbolas "degenerates into" the line-pair ECe, $E'Ce'$ and has for tangents every straight line through C; but the theoretical difference between these two views of the limit becomes apparent when we observe that the one makes the curvature at C *zero* whilst the other makes it *infinite.*

For some further discussion of these matters see the *Quarterly Journal of Mathematics* VIII. 126, 235, 343. X. 93; *Oxf. Camb. Dubl. Messenger of Mathematics* IV. 86, 129, 140, 148; Chasles *Sections Coniques* pp. 30—33; Salmon's *Higher Plane Curves* pp. 377, 383 (ed. 3, 1879).

HEXAGRAMMUM MYSTICUM.

PROPOSITION XIV.

126. *The three pairs of opposite sides of any hexagon inscribed in a conic have their intersections in one straight line.*

(i) Let $ABCDEF$ be any six points on a conic, and let O, P, Q be the intersections of (AB, DE), (BC, EF), (CD, AF).

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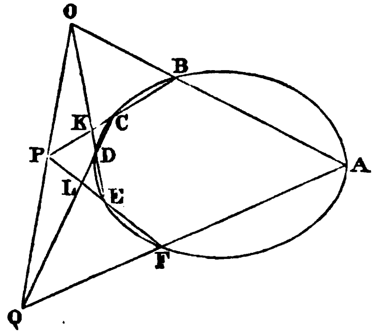
[*] This appears also by projecting the conic upon any plane from any vertex *in its own plane.*

[†] If λ, μ, ν be the perpendiculars from three points upon a straight line, the envelope of a line subject to the relation $\lambda\mu\nu = c^3$ assumes a corresponding line-form when c vanishes. By supposing each coordinate to become equal to a perpendicular of the triangle of reference whilst the product of the remaining two coordinates vanishes, we see that the limit of the curve is made up of three parts each of which constitutes a side of the triangle of reference or its complement. See also *Mathematical Questions from the* EDUCATIONAL TIMES, vol. XVI. 43.



Then since the four points $ACDE$ are equicross with respect to B and F, therefore

$$\{OKDE\} = B\{ACDE\} = F\{ACDE\} = (QCDL\},$$

if K be the intersection of BP, DO and L the intersection of FP, DQ.

And since the ranges $\{OKDE\}$ and $\{QCDL\}$ are thus equal and have a common point D, therefore the lines OQ, EL, KC meet in a point, or the points OPQ lie in a straight line,* as was to be proved.

(ii) Otherwise thus. Let O and Q be the intersections of AB, DE and AF, CD respectively. And let OQ meet AD in R, and BC in P, and EF in P'. Then will P' coincide with P.

For the points O, Q and the conic determine upon the transversal OQ an involution to which, by a property of the inscribed quadrilateral $ABCD$, the couple PR belong (Prop. x); and by a property of the quadrilateral $ADEF$ the couple $P'R$ belong to the same involution,† and therefore P' coalesces with P.

(iii) Otherwise thus.‡ Consider the surface generated by a straight line which always meets three fixed non-intersecting straight lines in space.

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* This line changes its position when the points $ABCDEF$ are taken in a different order. On the various Pascal-lines OPQ see the note on Pascal's theorem at the end of Salmon's *Conic Sections*; and see Townsend's *Modern Geometry* chap. 17.

† The proposition is thus virtually a corollary (Art 122 § i) from the theorem *Ad quatuor lineas*. See also Salmon's *Conic Sections*, Art. 267.

‡ This proof, as it stands, is taken from *Math. Questions from the* EDUCATIONAL TIMES XVIII. 83 (1873). For the corresponding proof of Prop. XV. see vol. XIX. 65. Both theorems had been treated in this way by DANDELIN in vol. III. of the *Nouveaux Mémoires de l'Académie &c. de Bruxelles* (1826).

Let 1, 2, 3, be the fixed lines and 1′, 2′, 3′ any three positions of the moving line. Then the common section of the two planes through 3′, 1 and 2, 2′ respectively passes through the points (3′, 2) and (1, 2′).

In like manner the common section of the planes through 1′, 2 and 3, 3′ passes through the points (1′, 3) and (2, 3′); and the common section of the planes through (2′, 3) and (1, 1′) passes through the points (2′, 1) and (3, 1′). The three common sections therefore form a triangle, and consequently *lie in one plane*.

Now let the surface be cut by any arbitrary plane. This plane will be met by the planes through 3′, 1; 2, 2′; &c. in a hexagon 1, 1′, 2, 2′, 3, 3′; and by the three common sections (since they are co-planar) in three *collinear* points *PQR*, which are also the intersections of the opposite sides of the hexagon. The proposition is thus true for any plane section of the ruled quadric, and therefore for any conic.

Corollary 1.

Five points *BCDEF* on a conic being given, we may now find any number of sixth points *A* on the curve, viz. by drawing arbitrary lines *BO* through one of the given points *B*, and then determining successively the points *OPQ*, and the line *QF*, and its intersection with *BO*. Notice that *A* is a vertex of the variable triangle *AOQ*, the extremities of whose side *OQ* slide along fixed lines *ED* and *CD*, whilst its three sides pass through three fixed points *PBF* respectively.

[Prop. I. Cor. 2.

Corollary 2.

If *ABCEF* be five given points on a conic, the *tangent* at any one of them *C* may be constructed by this proposition; for we have only to make *D* coincide with *C*, in which case the line *CQ* becomes the tangent at *C*. Again, by supposing *C* to coincide with *B* and *E* with *F*, we deduce that the tangents at the vertices B and F of a *quadrilateral* $ABDF$ inscribed in a conic intersect upon the straight line which joins the points of concourse of its sides AB, DF and AF, BD.

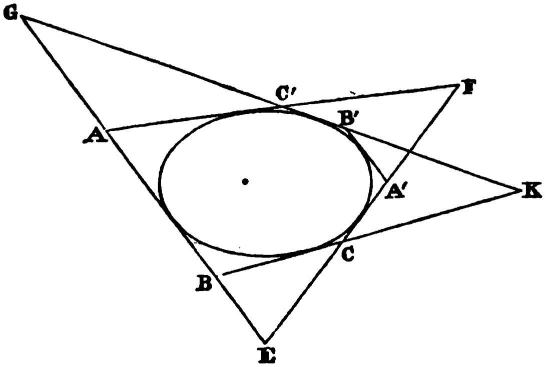
PROPOSITION XV.

127. *The joins of the three pairs of opposite vertices of any hexagon circumscribing a conic meet in a point.*

(i) Let the tangents at A, B, &c. in the preceding figure be a, b, &c.; and let ab denote the intersection of any two of them a and b.

Then the join of ab and de is the polar of O; the join of bc and ef is the polar of P; and the join of cd and fa is the polar of Q. And these three joins meet in a point, since their poles OPQ are in one straight line.

(ii) Otherwise thus. Let AA', BB', CC' be the opposite vertices of any hexagon circumscribing a conic; and let the four tangents AB, BC, $A'B'$, $C'A$ determine the range $\{ECA'F\}$ on the tangent CA', and the range $\{GKB'C'\}$ on the tangent $B'C'$.



These ranges being equal (Prop. I), we have

$$A\{BCA'C'\} = \{ECA'F\} = \{GKB'C'\} = B\{ACB'C'\};$$

and therefore, AB being common to the two pencils, their rays (AC, BC), (AA', BB'), (AC', BC') meet on a straight line (Art. 105), or the diagonals AA', BB', CC' of the hexagon meet in one point.

Corollary 1.

Having given five tangents to a conic we may determine their points of contact by this proposition; for if the summit C' of the circumscribing hexagon be *on the curve*, the tangents AC' and $B'C'$ being supposed to coalesce, then the line joining the opposite summit C to the intersection of AA' and BB' determines by its intersection with AB' the required point of contact C'. We may also determine an infinity of other tangents to a conic when five tangents AB, BC, CA', $A'B'$, $B'C'$ are given; for if we draw any line through the given point A' to meet AB in A, the point C' may be determined as above.

Corollary 2.

The orthocentre of any triangle is a point on the directrix of every parabola inscribed in it. For if abc be any three tangents to a parabola, $a'c'$ the tangents at right angles to a and c respectively, and ∞ the line at infinity, which together make up a hexagon $abcc'\infty a'$ circumscribing the parabola, then the joins of ab, $c'\infty$ and bc, $a'\infty$ are two of the perpendiculars of the triangle abc; and the join of the joins of the orthogonal tangents aa' and cc' is the directrix; and, by the proposition, these three joining lines cointersect.[*]

SCHOLIUM E.

PASCAL's theorem (Prop. XIV)—elsewhere called by him the theorem of the *Mystic Hexagram*—was enunciated without proof in his *Essais pour les Coniques* (1640) as a property of the circle, which might be generalised by projection, and then used as the foundation of a complete treatise on conics. See *Œuvres de Blaise Pascal*, IV. 1—6 (nouv ed. Paris, 1819); Chasles' *Aperçu Historique*, pp. 68—74.

BRIANCHON's theorem (Prop. XV) was deduced from Pascal's by means of Desargues' properties of what are now called polars (Scholium B). The author's proof of his theorem, given in his *Mémoire sur les Surfaces courbes du second Degré* (*Journal de l'Ecole polytechnique*, tome VI. 297—311, 1806), was as follows.

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[*] This proof is given, as by Mr. John C. Moore, in Salmon's *Conic Sections* (Art. 268, Ex. 3, sixth ed. 1879). See also Scholium p. 57. Brianchon and Poncelet had deduced the reciprocal theorem of Art. 69 from Pascal's hexagram.

Take any three concurrent straight lines PAA', PBB', PCC' in space, and let $LNMlnm$ denote the six intersections

$$(AB,\ A'B'),\ (BC,\ B'C'),\ (CA,\ C'A'),$$
$$(AB',\ A'B),\ (BC',\ B'C),\ (CA',\ C'A)$$

respectively. Then the four triads of points LMN, Lmn, lMn, lmN are evidently collinear, since they lie severally upon the common sections of the four pairs of planes

$$(ABC,\ A'B'C'),\ (ABC',\ A'B'C),\ (AB'C,\ A'BC'),\ (A'BC,\ AB'C').$$

And since every two of these triads have one point in common the four common sections and therefore the six points $LMNlmn$ lie in one plane, which also together with point P divides each of the segments AA', BB', CC' harmonically. [Art. 107.

If the whole figure be now projected orthogonally upon any plane, then (with the same notation) the six points $LMNlmn$ will in general still lie by threes upon four separate straight lines, in the order above-mentioned; but if any *other* three of them as LmN be also collinear the *six* points will then lie in one straight line, since the plane of projection must be at right angles to the plane of the original six points; and this line together with the point P will divide the segments AA', BB', CC' harmonically.

This is the case when AA', BB', CC' are concurrent chords of a conic, since their extremities may be taken in any order to form an inscribed hexagon (Prop. XIV). For example, the hexagon $ABC'A'B'C$ has for its Pascal-line LMn, on which the remaining three points lmN must also lie. Brianchon then observes that two of the three concurrent chords suffice to determine this line, whilst the third CC' may be supposed to turn about P, and to coincide with either of the former, or to become itself a tangent (if P be an external point). Having thus virtually given a fresh proof of the properties of polars,* he at once deduces his own theorem (Prop. XV.) from the reciprocal theorem of Pascal, which he takes from the *Géométrie de position* (Carnot), probably not knowing to whom it was due. See also Gergonne's *Annales* IV. 196, 379 (1813—14).

This brilliant application of Desargues' theory of polars, in conjunction with the property that *the polar planes of all points on one quadric with respect to a second envelope a third,* which Brianchon proved in the same article (as an extension from the case of similar and coaxal quadrics), served as a basis for the method of Reciprocal Polars, the full development of which was so largely due to PONCELET (Crelle's *Journal* IV. 1—71, 1829).

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* Pascal himself also had doubtless deduced the properties of polars (which he would have learned from Desargues) from his hexagram.

EXAMPLES.

696. If ABC, DEF be two triangles such that AD, BE, CF meet in a point, the intersections of (BC, EF), (CA, FD), (AB, DE) lie in one straight line, and conversely*; and every tetrad of radiants or collinear points in the figure is harmonic.

697. If the vertices of a triangle slide severally on three fixed radiants, and if two of its sides pass through fixed points, the third side passes through a third fixed point in a line with the former two, and conversely.

698. If one quadrilateral be divided into two others by any straight line, the diagonals of the three intersect in three collinear points.

699. Prove for the case of the circle that any four points on the curve and the tangents thereat are equicross; and that the cross ratio of any four points $ABCD$ on the curve is equal to $\frac{AB.CD}{AD.BC}$.

700. Prove that the sides and diagonals of a quadrilateral determine an involution on any transversal; and that its six summits subtend a pencil in involution at any point in its plane.

701. The circles on the three diagonals of a complete quadrilateral as diameters are coaxal; and they are orthogonal to the circle through the three intersections of its diagonals; and they determine an involution on any transversal.

702. Any two triangles which are reciprocal polars with respect to a circle† are in homology.

703. Find the locus of intersection of tangents to two given circles whose lengths are in a constant ratio.

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* Two such triangles are said to be *in perspective* or *in homology*. Solutions of Exx. 696–702, 705 are given in McDowell's *Exercises in Euclid and in Modern Geometry*, pp. 184–187, 227, 239 (new edit. 1878).

† The same may be proved for any conic, as (for example) in Cremona's *Éléments de Géométrie Projective*, p. 227 (1875).

704. The pairs of radiants from any point to the vertices of a triangle and parallel to its opposite sides respectively form a pencil in involution.

705. Deduce by reciprocation from the property of the orthocentre that, if from any point radiants be drawn to the summits of a triangle, the radiants at right angles to them meet the opposite sides of the triangle in three collinear points.

706. The nine-point circle N of the tetragon determined by a triad of points and their orthocentre (Art. 125, Cor. 3) touches the sixteen circles inscribed or escribed by fours to the four triangles determined by the summits of the tetragon (note, p. 191). If ABC and O be the points of contact of any of these sixteen circles with its triangle and with N respectively, the sixteen sets of lines OA, OB, OC, making in all forty-eight lines, pass by fours through the extremities of the six diameters of N parallel to the sides and diagonals of the tetragon; and every two tetrads which pass through opposite extremities of the same diameter have equal cross ratios.*

707. Prove by reciprocation from the theorem *Ad quatuor lineas* (or otherwise), that if a quadrilateral be circumscribed to a circle, the ratios of the products of the distances of its three pairs of opposite summits from any fifth tangent are invariable.†

708. From the anharmonic point-property of a conic deduce the theorem *Ad quatuor lineas*; and thence deduce the theorems of Art. 16, and the property of any principal or oblique ordinate. Shew also how to deduce the anharmonic property of four tangents from that of four points ‡

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* See Dr. Casey's article in the *Quarterly Journal of Mathematics*, IV. 245.

† See Mulcahy's *Principles of Modern Geometry*, p. 48 (ed. 2, 1862).

‡ All the chords PQ drawn to a conic from a given point P upon it are bisected by a similar conic touching the former at P and passing through its centre O. Let the tangent at Q meet that at P in R. Then OR meets PQ in a point q lying on the inner conic; and by the point-property of the latter,

$$P\{Q\} = P\{q\} = O\{q\} = O\{R\} = \{R\}.$$

This proof is from GASKIN'S *Geometrical Construction of a Conic Section* p. 26 (1852).

709. From the anharmonic tangent-property of a conic deduce the relation between the intercepts made upon a pair of parallel tangents by any third tangent.*

710. From the point-property of a conic, $A\,\{ABCD\} = B\,\{ABCD\}$, deduce that if from any point K on the chord AB a transversal be drawn meeting the tangents at A, B in T and T', and the conic in C and D, then

$$KC.KT'.TD = KD.TK.T'C.$$

711. Deduce elementary properties of the hyperbola from the relation.

$$\infty\,\{AB\infty\,\infty'\} = \infty'\,\{AB\infty\,\infty'\},$$

where ∞ and ∞' are its two points at infinity, and AB any other two points on the curve.

712. Deduce Art. 23, Cor. 3 from the relation

$$P\{PEQ\infty\} = \infty\,\{PEQ\infty\},$$

where ∞ is the point at infinity on the parabola.

713. Shew also by cross ratio that three fixed tangents to a parabola divide any fourth in a constant ratio.

714. Deduce from Prop. VI. that, if a conic touches the sides of a triangle and passes through the centre of its circumscribed circle, this circle touches the orthocycle (note, p. 280) of the conic.

715. Deduce from Prop. IX. that the nine-point circle of every triangle self-polar with respect to a parabola passes through the focus; and construct a triangle self-polar to every parabola inscribed in a given triangle.

716. If OP and OQ be tangents to a conic, the circle through P which touches OQ in Q is such that triangles self-polar with respect to the conic can be circumscribed to it.

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* CHASLES has founded his *Traité des Sections Coniques* upon the anharmonic properties of conics (cf. *Aperçu Historique*, pp. 39, 334–344). The properties of diameters and of the foci are deduced in chaps. VI. and X. The same general method is followed by Cremona; and it is given as an alternative by Rouché and De Comberousse (*Traité de Géométrie* § 1125, 4me ed. Paris 1879).

717. Shew how to inscribe in a given conic a triangle (or n-gon) whose sides pass severally through given points.[*]

718. Having proved the properties of polars by cross ratio, deduce the fundamental property of a diameter of a conic.

719. Prove Prop. II. by the same method, and deduce the elementary properties by which it was proved in the text.

720. From the properties of quadrilaterals inscribed or circumscribed to a conic, deduce that the diagonals of every inscribed parallelogram are diameters of the conic; and that supplemental chords are parallel to conjugate diameters; and that the diagonals of every circumscribed quadrilateral are conjugate diameters.

721. If a variable tangent to a conic meet the tangents from a given point L in F and H, and if M and K be a certain pair of fixed points on the fixed tangents; shew that $MF.KH$ is constant,[†] and deduce that a variable tangent to a conic divides any two fixed tangents homographically.

722. If $ABCDE$ be a pentagon circumscribing a parabola, the parallels from B to CD and from A to DE intersect upon CE.

723. If $ABCD$ be a quadrilateral circumscribing a parabola, the parallels from A to CD and from C to AD intersect on the diameter through B; and every other tangent divides AD and BC (or AB and CD) proportionally.[‡] Consider also the limiting case in which ABC is a straight line.

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[*] On Exx. 717 &c. see Salmon's *Conic Sections* §§297, 326–8, Exx., where Townsend's solution is given; Rouché et De Comberousse *Traité de Géométrie* § 1134. The problem—for a simple case of which see Pappus *Collect.* lib. VII, prop. 117—was solved by PONCELET, and analytically by GASKIN. See *Historical Notices respecting an Ancient Problem* in The Mathematician vol. III. pp. 75, 140, 225, 311, 42 (suppl.).

[†] See NEWTON's theorem Ex. 371, with Ex. 364, note; and compare Ex. 726, note. See also CHASLES *Géométrie Supérieure* § 120; *Sections Coniques* § 56.

[‡] Exx. 722–3 having been deduced from Brianchon's hexagon in Quetelet's *Correspondance mathématique et physique* IV. 155, Chasles was led (ibid. IV. 364, V. 289) from Ex. 723 to Ex. 724 (which is equivalent to the *anharmonic property* of four tangents to a conic), apparently without being aware that an equivalent theorem (Ex. 721, note) had been proved by NEWTON. See also *Aperçu Historique* pp. 341–4 (Note XVI.).

724. In a quadrilateral $ABCD$ circumscribed to a conic, the ratio of the ratios in which any fifth tangent divides either pair of its opposite sides is constant.

725. If the fifth tangent meet AB, CD in M and N, and if a sixth meet AD, BC in P and Q, then

$$AM.BQ.CN.DP = AP.DN.CQ.DM.$$

726. *Quatuor rectis BL, BI, DK, DH positione datis, ducere quintam LH talem, ut partes abscissæ HI, IK, KL sint in ratione data.*[*]

727. If a fixed conic S and a variable conic S' be inscribed in the same quadrilateral, the four points in which S' intersects S subtend at any point on S' a pencil whose cross ratio is constant, being equal to that of the range in which the sides of the quadrilateral meet any fifth tangent to S.[†]

728. If the tangent at O to a conic meet any other three tangents in the points abc, and meet their three chords of contact in $a'b'c'$, prove that $\{Oabc\} = \{Oa'b'c'\}$.

729. If AB be a given chord of a conic, and PQ a variable chord such $\{APQB\}$ is constant, the envelope of PQ is a conic touching the former at A and B.

730. If the chords AB and CD of a conic be conjugate, and ACB be a right angle, and a chord DP meet AB in Q; prove that the angle PCQ is bisected by CA or CB.

731. If ABC be a triangle circumscribing a parabola and abc the points at infinity on its sides, the tangents from

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[*] LAMBERT *Insigniores Orbitæ Cometarum Proprietates* sect. I. lemma 18 §§ 51–53 (1761). The envelope of LH is shewn to be the parabola touching the four given lines (Art. 28, Cor. 8). [The problem had been solved in another way in the *Arithmetica Universalis* prob. 52 (ed. 1707)—al. prob. 56]. Here we have obviously the anharmonic property of four tangents to a parabola; and by stating the result in the projective form that *the ratio of the ratios* $\frac{HI}{HL}$ *and* $\frac{KI}{KL}$ *is constant* we at once shew the property to be true for all conics. [See also the *Principia* lib. I. sect. V. lemma 27 Cor., where WREN and WALLIS are referred to for earlier solutions.]

[†] Briefly thus: the cross ratio of the common points of any two conics in the one is equal to that of their common tangents in the other.

any point O to the parabola belong to the involution $O\{Aa\,Bb\,Cc\}$. Hence shew that the directrix of every parabola inscribed in a triangle passes through its orthocentre. [Art. 110.

732. The joins of four points on a conic meet any transversal in three pairs of points in an involution, to which the intersections of the transversal with the conic also belong. Hence deduce (by removing the transversal to infinity) that every conic through a triad of points and their orthocentre is a rectangular hyperbola.

733. If AEB and CDF be two triads of collinear points, the intersections of (AF, CE), (BF, ED), (BC, DA) are in one straight line.*

734. In a hexagon inscribed in a conic, if two pairs of alternate sides are parallel the third pair are parallel.

735. In every hexagon inscribed in a conic the two triangles determined by the two sets of alternate sides are in homology. State the reciprocal theorem.

736. The Pascal lines of the sixty hexagons determined by a Pascal hexastigm pass by threes through twenty points; and the Brianchon points of the sixty hexagons determined by a Brianchon hexagram lie by threes on twenty straight lines.†

737. If two conics touch one another at A and B, and if LM be a chord of the outer which touches the inner conic; find the loci of the intersections of AL, BM and AM, BL.

738. The chords joining four points on a conic to any fifth P and to any sixth Q intersect in four points lying on a conic through P and Q.

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* PAPPUS *Collectio* lib. VII. prop. 139 (vol. II. p. 887, ed. Hultsch); Simson *De Porismatibus* p. 414; Chasles *Porismes* p. 77. Note that $AFBCED$ is a hexagon inscribed in a line-pair, so that Pascal's theorem is a generalisation of this lemma of Pappus.

† See TOWNSEND'S *Modern Geometry* II. 172. The terms hexastigm and hexagram are here very appropriately used to denote the figures determined by six *points* and *lines* respectively, taken in any order. In the text however I have retained the term hexagram as a designation of Pascal's figure out of regard for historical considerations.

739. If a conic S be inscribed in a triangle self-polar with respect to a conic S', shew that triangles self-polar with respect to S can be inscribed in S'.

740. Given the sum of the squares of the axes of a conic inscribed in a given triangle, the locus of its centre is a circle concentric with the polar circle of the triangle.*

741. Given five points on a conic, find (by cross ratio or involution) its second intersection with any straight line through one of the five points, and its two intersections with any other straight line; and determine its points at infinity and its asymptotes, real or imaginary.

742. Prove by cross ratio that five tangents determine a conic; and determine other tangents and their points of contact; and shew how to construct the tangents from any given point, real or imaginary.

743. Prove by involution that if three sides of a quadrilateral inscribed in a conic turn about three points in a straight line, the fourth side turns about a point in the same straight line; and hence shew how to inscribe in a conic a triangle whose three sides pass severally through three collinear points.

744. Prove Carnot's theorem, that if aa', bb', cc', be the three pairs of points in which a conic meets the sides BC, CA, AB of a triangle, then

$$Ab.Ab'.Bc.Bc'.Ca.Ca' = Ac.Ac'.Ba.Ba'.Cb.Cb'.\dagger$$

Prove also that the same relation subsists when A, B, C denote the *sides* of a triangle; a, a', &c. the *tangents* from its vertices to a conic; and Ab denotes the *sine* of the angle between any two lines A and b.

745. The distances pqr of any point on (or tangent to) a given conic from three fixed lines (or points) are connected

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* See the *Quarterly Journal of Mathematics* x. 130.

† This is an obvious corollary from Art. 16. It is given in Carnot's *Géométrie de position* § 236 (Paris, 1803) as a case of a more general theorem.

by a relation of the form

$$Pp^2 + Qq^2 + Rr^2 + P'qr + Q'rp + R'pq = 0,$$

where P, Q, &c. are constant coefficients.

746. If four tangents to a conic parallel to four chords $abcd$ through either focus meet any fifth tangent in points $ABCD$, then

$$\{ABCD\} = \{abcd\} \frac{pp'}{qq'},$$

where p, p' and q, q' are the perpendiculars upon the fifth tangent from two pairs of opposite intersections of the four tangents. If the latter be fixed pp' varies as qq'. Hence deduce that the product of the focal perpendiculars upon any tangent to a conic is constant.*

747. If three summits of a quadrilateral circumscribing a conic slide severally on three rays of a pencil, the fourth slides on a fourth ray. Hence shew how to circumscribe to a conic a triangle whose three vertices lie on three given radiants.

748. If upon a given arc AB of a circle whose centre is O there be taken any arc Am, and likewise an arc Bn equal to $2Am$, then $O\{m\} = B\{n\}$. Hence deduce a solution of the problem, to trisect a given angle AOB. [Ex. 528.

749. The product of the distances of any point on a hyperbola from a given pair of parallels to the asymptotes varies as its distance from the chord intercepted by the parallels: and the product of the distances of any point on a parabola from two fixed diameters varies as its distance from the chord joining their extremities.

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* The distances of any two tangents from either focoid (Art. 123, Cor. 4) being in a ratio of equality, the products of the focal perpendiculars upon any two tangents are in a ratio of equality. The cross ratio of the range in which any tangent meets the sides of the quadrilateral $S\phi H\phi'$ is equal to $\frac{-4SY.HZ}{SH^2}$, where SY and HZ are the focal perpendiculars upon the tangent (*Oxf. Camb. Dubl. Messenger of Mathematics* IV. 94). Chasles calls the points of concourse of common tangents to two conics "points ombilicaux" (*Sections Coniques* chap. XIV.), with reference to the use of the term *Umbilicus* for focus noticed above on p. 5.

750. From Ex. 744 deduce a construction for a conic passing through four given points and touching a given straight line: and shew that the lines joining the vertices of a triangle circumscribing a conic to the opposite points of contact cointersect: and when four points on a conic and the tangent at one of them are given, shew how to draw the osculating circle at that point.

751. Through the centre of a conic and any conjugate triad with respect to it a hyperbola can be described having its asymptotes parallel to any given pair of conjugate diameters.

752. The system of radiants from any point parallel to the tangents to a parabola is homographic with the range in which these tangents meet any fixed tangent.*

753. If from a series of collinear points pairs of perpendiculars be drawn to two fixed straight lines, the joins of the feet of the several pairs of perpendiculars envelope a parabola touching the two fixed lines.

754. If any chord of a conic drawn from a fixed point O upon it meets the sides of a given inscribed triangle in points ABC and the conic again in P, shew that $\{ABCP\}$ is constant; and deduce a construction for the tangent at a given point to a conic of which four other points are likewise given.

755. If ABC be the intersections and abc the points of contact of three fixed tangents to a conic, the product of the distances of any tangent from A and a varies as the product of its distances from B and C: the product of its distances from b and c varies as the square of its distance from A: the pairs of radiants from any point O to BC and Aa determine an involution to which the tangents from O to the conic belong: and these tangents with Ob and Oc determine an involution having OA for one of its double rays.

756. Deduce from Brianchon's hexagon that when a quadrilateral circumscribes a conic the joins of its opposite points of

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* For solutions of Exx. 741–759, 765–800 see Chasles' *Sections Coniques* pp. 8–67, 72–109, 137—145, 160, 204, 209, 241—299, 321 &c.

contact pass through the intersection of its two diagonals; and that in a triangle circumscribing a conic, the three lines joining its vertices to the opposite points of contact meet in a point.

757. The three pairs of lines from the vertices of a triangle to the intersections of its opposite sides with a conic are tangents to one conic: and the lines from any two points to the vertices of a triangle meet its opposite sides in six points lying on one conic. Reciprocate these two theorems; and from the second of them deduce the property of the nine-point circle.

758. The ratio of the products of the distances of any point on a conic from the odd and even sides respectively of a given inscribed $2n$-gon is constant: and the products of the intercepts on any chord made by the odd and even sides are in the same ratio from whichever extremity of the chord the intercepts are measured.

759. The ratio of the products of the distances of any tangent to a conic from the odd and even summits respectively of a given circumscribed $2n$-gon is constant; and the ratio of the products of its distances from the summits and from the points of contact of any given circumscribed n-gon is constant.

760. If two angles of given magnitudes PAD and PBD turn about A and B as poles given in position, then if the intersection P of one pair of their arms be made to describe a conic, the intersection D of the other pair will in general describe a curve of the fourth order, having double points at A and B and at the limiting position of D when the angles BAP and ABP vanish together: but the locus of D will be of the third order if the angles BAD and ABD vanish together. If P describes a conic passing through A, then D describes a cubic having a double point at A and passing through B.* This cubic

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* This is NEWTON'S *Curvarum Descriptio Organica* (note p. 264). The case at the end of Ex. 760 follows from the principle that a cubic proper cannot have two double points (Salmon's *Higher Plane Curves* § 42). This special case is given by Chasles (*Aperçu historique*, p. 337) as a *generalisation* of NEWTON'S construction in the *Principia*.

degenerates into the line AB and a conic through A and B in the case in which the original conic passes through both A and B.

761. The nine-point circle of a triangle touches its inscribed and escribed circles at points lying on the ellipse which touches the sides of the triangle at their middle points.*

762. Reciprocate Maclaurin's description of a conic given in Art. 113, Cor. 2.

763. The sides of a quadrilateral inscribed in a conic meet the tangents at its opposite angles in four pairs of points lying on one conic.†

764. If a quadrilateral be circumscribed to a conic, the four pairs of lines joining its vertices to the opposite points of contact touch one conic.

765. If the sides of an n-gon turn severally about fixed points, whilst $n-1$ of its summits slide each on a fixed line; the n^{th} summit describes a conic passing through the fixed points on the two adjacent sides.

766. Shew also that any two sides not adjacent intersect on a fixed conic through the points about which they turn.

767. If the arms A and B of an angle pass each through a fixed point, whilst its summit slides on a fixed line; shew that the join of the points in which A meets one fixed line and B another envelopes a conic touching the join of the fixed points.

768. If each summit of an n-gon slides on a fixed line, whilst $n-1$ of its sides pass severally through (or subtend given angles at) fixed points; the n^{th} side envelopes a conic touching the lines on which its extremities slide; and every diagonal of the n-gon envelopes a conic.

769. Any two pairs of conjugate lines from a point O to a conic determine an involution whose double rays are the tangents from O to the conic. State the reciprocal theorem.

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* See Salmon's *Conic Sections*, § 345, Exx.

† Möbius *Barycentrische Calcul* § 281.

770. One point and one only of every conjugate triad with respect to a conic lies within the conic; and two sides of every self-polar triangle meet the conic.

771. The lines drawn from any point on a conic to two conjugate points *A* and *B* meet the conic at the extremities of a chord which passes through the pole of *AB*. State the reciprocal theorem.

772. If a quadrilateral be circumscribed to a conic, the extremities of any chord through the intersection of two of its diagonals lie on a conic passing through the extremities of both.

773. Any three pairs of points which divide the three diagonals of a quadrilateral harmonically lie on one conic.

774. If the extremities of two diagonals of a quadrilateral be conjugate points with respect to a conic, the extremities of the third will be likewise conjugate.

775. If two of the three pairs of joins of four points be conjugate lines with respect to a conic, the third pair will be conjugate with respect to it.

776. The pairs of chords drawn from a fixed point on a conic so as to make equal angles with a given line intercept a variable chord which passes through a fixed point.

777. The pairs of tangents to a conic from points on a straight line determine an involution on any transversal through its pole, or on any tangent to the conic.

778. The pairs of tangents to a parabola from points in the same straight line are parallel to conjugate rays of a pencil in involution.

779. Two tangents being drawn to a conic from any point on a fixed straight line, if x and x' be their distances from its pole, and y and y' their distances from a fixed point, shew that

$$\frac{x}{y} \pm \frac{x'}{y'} = \text{a constant.}$$

780. Any two ranges in involution on the same axis have one segment in common.

781. The locus of the middle point of a chord drawn from a fixed point to a conic is a conic through the point, and through the points of contact of the tangents from it to the original conic, and through the two points at infinity on that conic.

782. Find the envelope of a line which meets two fixed lines in a pair of conjugate points with respect to a given conic.

783. The envelope of the parallel from any point on a fixed straight line to the polar of the point with respect to a conic is a parabola touching the fixed line.

784. The locus of the intersection of a pair of conjugate lines with respect to a given conic, drawn each through a fixed point, is a conic, which passes through the two fixed points, and through the points of contact of the tangents from them to the original conic.

785. If two angles be circumscribed to a conic their two summits and their four points of contact lie on one conic.

786. Any transversal being drawn to a conic from a fixed point O, the perpendicular to it from its pole envelopes a parabola, which touches the polar of O and the tangents to the conic at the feet of the normals to it from O.

787. Circumscribe to a given conic a polygon having each of its summits upon a given straight line.

788. The poles of a given straight line L with respect to the system of conics through four given points is a conic, which with the line L divides the six joins of the four points harmonically, and passes through their three intersections, and through the two points on L which are conjugate with respect to every conic of the system: it also touches the sixteen conics which pass through the said conjugate points and touch by fours the sides of the four triangles determined by the given points.

789. If four conics pass through the same four points, the polars of any point with respect to them form a pencil whose cross ratio is constant, being equal to that of the tangents to the four conics at any one of their points of concourse: and reciprocally, if four conics touch the same four lines, the poles of any straight line with respect to them form a range whose cross ratio is constant, being equal to that of the points in which the four conics touch any one of their common tangents.

790. If two conics osculate at O, their tangents at the further extremities of any chord through O intersect on the tangent at O, and conversely: and every two equal and coaxal parabolas osculate at infinity.

791. Two conics which have two pairs of conjugate diameters of the one parallel to two pairs of conjugate diameters of the other must be similar and similarly situated.

792. Deduce from Art. 114 (i) that parallel conics* have a common chord (real or imaginary) at infinity; and if also concentric they have double contact at infinity. Shew how to draw a conic which shall be parallel to a given conic, and shall also pass through three given points or touch three given lines.

793. Three fixed conics having four points in common, shew that if a variable pair of transversals be drawn from fixed points O and ω to meet the three conics in triads of points mAB and mab respectively, the ratio of the ratios $\frac{OA \, . \, OB}{mA \, . \, mB}$ and $\frac{\omega a \, . \, \omega b}{ma \, . \, mb}$ is constant.† Hence deduce that a conic may be regarded as the locus of a point the square of the tangent from which to a fixed circle varies as the product of its distances from two fixed lines, which are common chords of the conic and the circle.

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* Similar and similarly situated conics may be called *parallel* since their curves are everywhere parallel at corresponding points: they have also been called "homothetic" (Chasles *Sections Coniques* § 378), which should rather mean "placed together." For another use of the term parallel see Gergonne's *Annales* XII. 1.

† Exx. 793 &c. have been extended to quadrics by Mr. Martin Gardiner in the *Quarterly Journal of Mathematics* X. 132—147.

794. If from any point on one of three conics which have four points in common a tangent be drawn to each of the remaining two, the ratio of the ratios of these tangents to the parallel diameters of their conics is constant: and if OPQ be the middle points of the intercepts made by the three conics on any transversal, then OP and OQ are in the ratio of the parallel focal chords of the second and third conics.

795. Four fixed conics having four points in common being met by a variable transversal, viz. two of them in the pairs of points aa' and bb', and the third in two points of which m is one, and the fourth in two points of which n is one; shew that the ratio of the ratios $\frac{ma . ma'}{mb . mb'}$ and $\frac{na . na'}{nb . nb'}$ is constant.

796. If $ABCD$ be four conics such that the eight points of concourse of AB and CD lie on one conic, the eight points of concourse of AC and BD (or AD and BC) lie on one conic.

797. When a point O has the same polar with respect to three conics ABC, three pairs of the common chords of AB, BC, CA respectively pass through O and form a pencil in involution: and when two conics A and B have each double contact with a third conic C, a pair of the common chords of A and B are harmonic conjugates with respect to their chords of double contact with C.

798. The common tangents to three conics taken in pairs form three quadrilaterals: shew that the three parabolas inscribed in them have a common circumscribed triangle.

799. If through the intersections of two given conics A and B a third conic C be drawn, and if from any point O on C there be drawn tangents Oa, Oa' to A and Ob, Ob' to B; the lines ab, ab', $a'b$, $a'b'$ and the four common tangents of A and B touch a fourth conic.

800. The locus of the point the pairs of tangents from which to two given conics form a harmonic pencil is a third conic, on which lie the eight points in which the given conics touch their common tangents. State the reciprocal theorem.

CHAPTER XI.

CONICAL PROJECTION.

128. Two figures A and B in any two planes are said to be in *Perspective* when a point O can be found in space such that every radiant from it to a point on A passes through a point on B, and conversely. Either figure is then said to be the *Central* or *Conical Projection* of the other on the plane of the former, the point O being called the *Vertex* or the *Centre* of projection. When O is at infinity the projection becomes parallel or orthogonal. [Art. 86.

Let P and Q be any two points in the plane of A, and P' and Q' their projections from the vertex O upon the plane of B. Then evidently the lines PQ and $P'Q'$ intersect upon a fixed straight line, viz. the common section of the planes of A and B. Now by projecting the whole figure orthogonally upon any one plane, or by supposing the planes of A and B to become coincident, we see that if to every point P of one figure corresponds a single point P' of another figure in the same plane, and conversely, and if PP' passes through a fixed point; then every line PQ in the one figure meets the corresponding line $P'Q'$ in the other upon a fixed straight line. For example, if the joins of the vertices of two triangles meet in a point, the joins of their opposite sides lie in one straight line. [Ex. 696.

Two figures thus related in one plane are said to be in *Perspective* or in *Homology*. We shall in general use the former term for this kind of correspondence, and the term *Projection* for the case of figures in perspective in space. The terms *Reversion* and *Homographic Transformation* will be explained in their place.

129. It is evident that Art. 88 applies to central as well as to parallel projection. Parallel lines however do not project from any vertex into parallels, except in the case in which they are parallel to the common section of the primitive plane with the plane of projection.

If through the vertex V of projection (fig. p. 314) there be drawn the plane Vab parallel to the plane of projection $AO'B$, and so as to meet the plane of the figure to be projected in the line ab; it is evident that all points on ab will be projected to infinity, or in other words they will remain unprojected. For this reason ab is called the *Unprojected Line*, and it is also said to be projected *to infinity*.

Since every point at infinity in the plane $AO'B$ corresponds projectively to some point on ab, we come again to the conclusion that all points at infinity in one plane lie in a straight line (Art. 17 Cor. 2). *The straight line at infinity is to be regarded as parallel to every other straight line in its plane*, since it intersects every such line at infinity: it is in fact coincident with the *circle of infinite radius** described about any point whatever in its plane. The line at infinity and the two *focoids* (Art. 123 Cor. 4) or *circular points* at infinity—so called because every circle in their plane passes through them—will be seen to be of peculiar importance in the projection and transformation of curves.

THE FOCOIDS.†

PROPOSITION I.

130. *Every circle in a given plane passes through the focoids, and conversely; and every two concentric circles in the same plane touch one another at the focoids.*

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* For a tangential equation to this circle, which is sometimes inadequately said to represent the focoids only, see Whitworth's *Trilinear Coordinates &c* Art. 382 (Cambridge 1866).

† This term is open to the objection that it combines a Latin word with a Greek ending: but we may perhaps be allowed to treat both as naturalised English expressions. In speaking of *the* focoids &c. we tacitly refer to a specified plane. Every plane not at infinity has its *two* focoids and its *one* line at infinity.

(i) Any number of right angles turning about their summits in one plane generate similar pencils in involution (Art. 110), whose imaginary double rays form two sets of parallels; that is to say, each set pass through one of two fixed imaginary points ϕ, ϕ' on the line at infinity.

These are accordingly the *foci* of the involution which the arms of all right angles in one plane determine upon the line at infinity in that plane; and conversely *every right angle AOB is divided harmonically by the lines $O\phi$ and $O\phi'$.*

If therefore O be a variable point at which a fixed line AB subtends a right angle, it follows from the harmonicism of $O\{A\phi B\phi'\}$ that the locus of O is a conic through the points $AB\phi\phi'$ (Art. 113). That is to say, every *circle* AOB passes through the focoids, and conversely every conic through the focoids is a circle.

(ii) The centre C being the pole of the line at infinity (which is the join of the focoids), it follows that the lines from C to the focoids touch the circle at those points; and hence that all circles in one plane which have any point C for their common centre touch one another at the focoids of that plane.

(iii) Or by §i and Art. 122 Cor. 5, all concentric circles in one plane touch one another at the focoids. This also follows from the consideration that any two diameters CX and CY of a circle which are at right angles are *conjugate lines* with respect to the circle, and the lines $C\phi$ and $C\phi'$* with respect to which they are harmonic conjugates must therefore touch the circle—and all circles having C for centre must touch one another—at ϕ and ϕ'.

Corollary.

Every rectangular hyperbola has for a pair of conjugate points with respect to it the focoids of its plane, since its points at infinity lie on two straight lines at right angles.

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* These lines may be regarded as the *asymptotes* of the circle (Art. 114).

PROPOSITION II.

131. *Every conic may be regarded as inscribed in the quadrilateral which has for opposite summits the real and imaginary foci of the curve and the focoids, and for diagonals the two axes of the conic and the line at infinity.*

(i) This is proved by the method of Art. 123 Cor. 4, where S and H may be either the real or the imaginary foci.
[Scholium A.

(ii) Otherwise thus. Every two lines through S which are conjugate with respect to the conic being at right angles (Art. 7), the lines $S\phi$ and $S\phi'$ which divide them harmonically are tangents to the conic (Art. 116 Cor. 2). That is to say, the lines joining the real or imaginary foci to the focoids touch the conic, as was to be proved.

PROPOSITION III.

132. *Any two straight lines drawn at a given angle in a given plane and the lines joining their point of concourse to the focoids form a pencil of constant cross ratio.*

For if ab be a fixed straight line, and ω any point at which it subtends an angle of given magnitude α, then by a property of the circle ωab, the pencil subtended by ab and the focoids at ω is of constant cross ratio; and the rays ωa, ωb may be parallel to any two lines OA and OB inclined at an angle α in the same plane.

Corollary.

Any plane figure may be moved about in any way in its own plane without changing its relation to the focoids, since every angle in the figure has an invariable relation to the focoids.

SCHOLIUM A.

Desargues regarded the opposite extremities of an infinite line as coincident or consecutive points, and the asymptotes of a hyperbola as its tangents at infinity (Poudra's *Œuvres de Desargues* I. 103, 197, 210, 245). Hence we deduce (Scholium B, p. 153) that the hyperbola is a single curve, which spreads completely across its plane *without breach of continuity*. It follows logically that no transversal can be drawn in the same plane so as not to meet the hyperbola. Nevertheless it is obvious that some lines—its conjugate axis for example—do not (however far produced) meet the curve in geometrical points. Thus we are driven to the conception of ideal or *imaginary* points and chords of intersection, and are led to say that every straight line meets any hyperbola (or other conic) in two points real or *imaginary*, which coalesce in the case of tangency. Although this is here given merely by way of inference, the words of Desargues himself (used in another connexion) are very appropriate to this subject: "*L'entendement ne peut comprendre comment sont les proprietez que le raisonnement luy en fait conclure*" (*Œuvres* I. 195).

BOSCOVICH has a very remarkable appendix to his treatise on conics, entitled *De Transformatione Locorum Geometricorum, ubi de Continuitatis lege ac de quibusdam Infiniti mysteriis* (Universæ Matheseos Elementa, tom. III. pp. 228—356), in which he brings out clearly and with an abundance of geometrical illustration the notions of positive and *negative* in direction: of geometrical *continuity*: of the transition from positive to negative through zero or *infinity*: of the *imaginary* chords of the hyperbola, whose squares are negative: and of the *quasi-elliptic* nature of the hyperbola, certain of the properties of which follow from properties of the ellipse by change of sign (§§ 678, 715, 758, 770, 808, 812, &c.). See also Scholium C, p. 101.

The discussion of these matters having been revived in the present century (Chasles *Rapport sur les progrès de la Géométrie* chap. I. § 19, p. 60), PONCELET at length worked out his theory of *cordes idéales* (1820); and he shewed that all circles in one plane pass through the same two imaginary points ϕ and ϕ' on the line at infinity, and that a focus S common to any two conics in one plane is a "centre of homology" or intersection of common tangents to the two conics. Hence it follows, by supposing one of the two conics to become a circle, that $S\phi$ and $S\phi'$ are tangents to every conic of which S is a focus. See Gergonne's *Annales* XI. 73, XII. 234; Poncelet *Traité des Propriétés Projectives des Figures* §§ 89—98, 258, 367, 453 (Paris, 1822). Plücker extended this conception to plane curves of all orders, regarding as a "focus" of any curve the point of concourse of any two tangents drawn to it from the focoids, one from each (Crelle's *Journal* x. 84—91; Salmon's *Higher Plane Curves* § 138).

According to Plücker's definition, the tangents from the focoids ϕ and ϕ' to an ellipse (or other conic) determine by their opposite intersections *two pairs* of "foci." If S be any one of the four, every pair of conjugate lines from S to the conic form a harmonic pencil with $S\phi$ and $S\phi'$ (Art. 116, Cor. 2), and are therefore *at right angles*. This, which is of course a corollary from Desargues' theory of polars was proved for the real foci by De la Hire (*Sectiones Conicæ* Lib. VIII. prop. 23, p. 189. Paris, 1685). The two points on the transverse axis at distance $\pm\sqrt{(CA^2 - CB^2)}$ from the centre C have been shewn to possess the property in question (Art. 7, Cor.);

and by symmetry, the two points on the conjugate axis at distance $\pm\sqrt{(CB^2 - CA^2)}$ from C must be the remaining two (or imaginary) foci. In order to prove that a point S is a focus of a given conic, it suffices to prove that TWO ***pairs of conjugate lines at right angles can be drawn to the conic from** S.* [Art. 110.

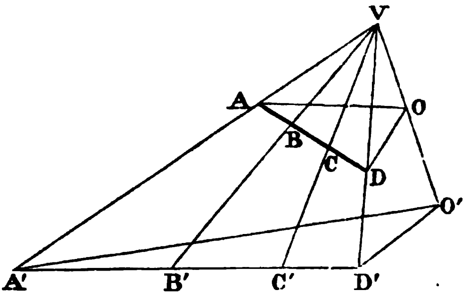
PROJECTION.

PROPOSITION IV.

133. *All rows of points and pencils of rays are homographic with their projections.*

(i) For if $ABCD$ be any row of four points in the primitive plane, and $A'B'C'D'$ their projections from a vertex V upon any other plane, it is evident that $\{A'B'C'D'\} = \{ABCD\}$. And if O be any fifth point in the primitive plane and O' its projection, then

$$O'\{A'B'C'D'\} = \{A'B'C'D'\} = \{ABCD\} = O\{ABCD\},$$



Thus every tetrad of radiants OA, OB, OC, OD or of collinear points $ABCD$ is equicross with its projection; a result which may be briefly expressed by saying that *figures in perspective are homographic.*

(ii). More generally,* let the joins of any number (say six) of points $ABCDEF$ be connected by a homogeneous and symmetrical relation

$$l.AB.CD.EF + m.AC.BE.DF + n.AD.BE.CF = 0,$$

in which the terms differ from one another only in their coefficients and in the *order* in which the letters $ABCDEF$ occur in them. And first let all the points lie in one straight line, and let VP be the perpendicular upon it from the vertex of projection.

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* See Salmon's *Conic Sections*, Art. 351.

Then since

$$AB = \frac{VA \, . \, VB}{VP} \, . \sin AVB; \quad CD = \frac{VC \, . \, VD}{VP} \, . \sin CVD; \ \&c.$$

the above relation reduces, by the omission of a common factor, to a relation between the sines of the angles which the joins of the six points subtend at V. It therefore still subsists when the points in question are replaced by their projections upon any plane.

And further, if any number of points $ABCDEF$ &c. lie on different straight lines, the perpendiculars upon which from V are VP, VP', VP'', &c., then any symmetrical and homogeneous relation between the joins of the points will still be projective, provided that it implicitly involves in every term the *same* factor $\frac{VA.VB.VC.VD.\&c.}{VP.VP'.VP''.\&c.}$. Thus Carnot's theorem (Ex. 744) is projective, so that when proved for the circle it is may be extended to all conics by projection.

Corollary.

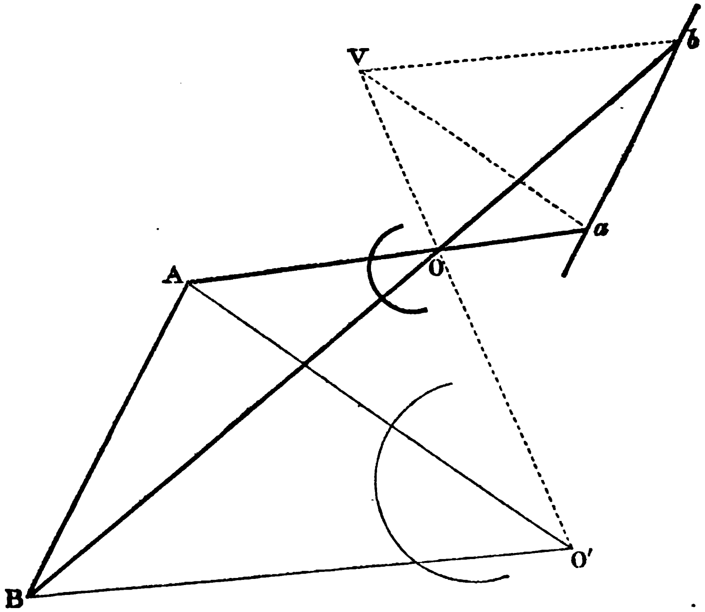
The properties of harmonic section, of poles and polars, and of involution are projective; so that it suffices to prove them for the simplest figure into which any figure to which they belong can be projected.

PROPOSITION V.

134. *Any straight line in the primitive plane can be projected to infinity, and any two angles in that plane can at the same time be projected into angles of given magnitudes.*

(i) Draw any straight line ab in the primitive plane, and take any plane Vab through ab for the "vertex-plane," in which the vertex V of projection is to lie. Then it is evident that the line ab projects to infinity upon any assumed plane of projection ABC' parallel to the vertex-plane.

(ii) Two conditions now suffice to fix the position of V in the vertex-plane.



To project a given angle AOB in the primitive plane into an angle of given magnitude α, let the arms of AOB meet the unprojected line in a and b; and upon ab describe in the vertex-plane a circular segment aVb containing an angle equal to α. Then the vertex V may be taken at any point on this segment.

For the vertex-plane and the plane of projection (being parallel) are met by the plane VOa in parallel lines Va and AO', and by the plane VOb in parallel lines Vb and BO'. Therefore, O' being the projection of O,

$$\angle AO'B = aVb = \alpha,$$

or the projection $AO'B$ of the angle AOB is of the assigned magnitude α.

To project a second given angle in the primitive plane into an angle of given magnitude β, let its arms meet the unprojected line in a' and b': then the vertex V must lie also on a segment described upon $a'b'$ in the vertex-plane so as to contain an angle equal to β: and the intersection of this with the segment on ab completely determines the position of V.

Corollary 1.

Project any four collinear points $ABCD$ into points $abcd$. Then in the special case in which one of the latter d is at infinity,

$$\{ABCD\} = \{abcd\} = \frac{ab.c\infty}{a\infty.cb} = \frac{ab}{cb}.$$

If therefore we determine the point D on a given straight line ABC so that $\{ABCD\}$ may be equal to a given ratio, and if any straight line through D be taken as the unprojected line, the projections of AB and CB will be in the given ratio. In like manner a second point D' on the unprojected line is determined by the condition that the segments of a second line $A'B'C'$ shall project in another given ratio.

Corollary 2.

Any pencil of rays in involution may be projected into a rectangular pencil in involution by projecting the angles between any *two pairs* of its conjugate rays into right angles. [Art. 110.

Corollary 3.

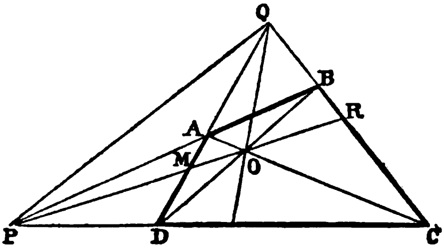
Any two points F and F' may be projected into the focoids of a given plane. For if AB and CD be any two segments in the involution of which F and F' are the foci, we have only to project the line FF' to infinity and any two angles AOB and CPD in the primitive plane into right angles (Art. 130 § i). This construction is *imaginary* when F and F' are real points.

PROPOSITION VI.

135. *Any quadrilateral may be projected into any other quadrilateral of given form and magnitude.*

(i) To project a given quadrilateral $ABCD$ into a square, project one of its angles BAD and the angle AOD between its two diagonals into right angles, and its third diagonal PQ to infinity. Thus the projection of $ABCD$ becomes a square, whose magnitude is determined by the distance of its plane from the vertex-plane.

(ii) To project a given quadrilateral $ABCD$ into another of given form, it suffices to project one of its angles BAD and the angle AOD between its two diagonals into angles of certain given magnitudes, and the segments AO, OC and BO, OD into segments which are in certain given ratios.



[Art. 134 Cor. 1.

For in the projection—the same letters being used—if AO be taken arbitrarily, the point C is determined by the ratio $\frac{AO}{OC}$; and the position of the line BOD is known; and from the angle BAD and the ratio $\frac{BO}{OD}$ the points B and D are determined. The form of the projection being thus determined, its magnitude may be increased or diminished at pleasure by moving the plane of projection towards or away from the vertex-plane.

Corollary.

Any four points or lines in one plane may be projected into any other four points or lines in one plane.

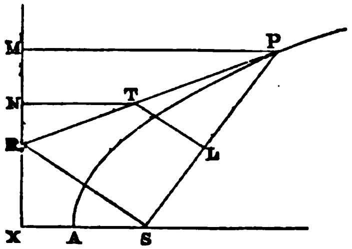
PROPOSITION VII.

136. *A given conic may be projected into a conic having the projections of two given points for foci, or the one for centre and the other for a focus.*

(i) To project a given conic Q and a given point S within it into Q' and S' respectively so that S' may be a focus of Q': draw from S any two pairs of lines conjugate with respect to Q, and project the angles contained by them into right angles. Thus S' becomes a focus of Q', being a point such that *every* pair of conjugate lines drawn from it to Q' are at right angles. [Art. 134 Cor. 2.

We may at the same time project a given point C in the plane of Q into the centre C' of Q', viz. by taking the polar of C with respect to Q for the unprojected line.

(ii) Otherwise thus. Let CS and the tangent at any assumed point P to the conic Q meet the polar of S in X and R respectively. Then if the polar of C be projected to infinity and each of the angles RXS and RSP into a right angle, the points C and S will be projected into a centre and focus of Q', as before.



(iii) By properly choosing the point C in the foregoing constructions, we may project Q so that any two points S and H within it project into S' and H' the real foci of Q'.

For if SH meets Q in A and B, and if the double points of the involution determined by the couples AB and SH be the point C on SH and the point G on its complement; then in the projection, the double point C' bisects *every segment* $S'H'$, $A'B'$, &c. of its involution, since in conjunction with the second double point (in this case at infinity) it divides every such segment harmonically. [Art. 112.

Hence S' and H' are equidistant from the centre C' of Q', and since S' is a focus H' is likewise a focus, as required.

(iv) *The system of conics inscribed in a given quadrilateral* $SFHF'$ *may be projected into confocal conics* by projecting F and

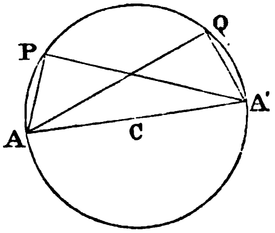
F' into the focoids of the plane of projection (Art. 134 Cor. 3). This construction is *imaginary* when the quadrilateral $FSHF'$ is real: but the foregoing constructions are always real, the points S and H being taken within Q.

PROPOSITION VIII.

137. *A given conic may be projected into a circle having the projection of a given point for centre: a system of conics through four given points may be projected into coaxal circles: or a system of conics touching one another at two given points into concentric circles.*

(i) By taking the point C at S in Art. 136 we project the given conic into a conic having the same point (not at infinity) for both centre and focus; that is to say, we project it into a circle having the projection of a given point for centre.

(ii) Otherwise thus. Take the polar of any point C for the unprojected line: through C draw any chord ACA', and



project the angles which it subtends at two assumed points P and Q on the given conic into right angles. Then in the projection, the same letters being used,

$$CA = CA' = CP = CQ,$$

or the projection is a circle about C as centre.

Thus the *two angles* determine the species of the projection, and the unprojected line may be taken arbitrarily.[*]

(iii) Hence, by projecting any conic into a circle and one of its chords FF' into the line at infinity, we may project any two points F' and F'' on a conic into the focoids, as was otherwise shewn in Art. 134 Cor. 3.

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[*] This may also be deduced from a consideration of the circular sections of a cone described arbitrarily on any given conic as base (Salmon's *Conic Sections* Art. 365).

It follows that all conics through two given points may be projected into circles, and all conics through four given points into *coaxal circles*, and all conics touching one another at two given points into *concentric circles*. [Art. 130 § ii.

PROPOSITION IX.

138. *The arms of any angle of constant magnitude in a given plane may be projected into rays of a pencil of constant cross ratio, whose other two rays pass each through a fixed point.*

For the arms of a constant angle and the lines joining their intersection to the focoids form a pencil of constant cross ratio (Prop. III.), which projects upon any plane into a pencil of constant cross ratio, two of whose rays pass through the projections of the focoids. Note that this pencil is *harmonic* when the constant angle is a *right angle*. [Art. 130 § i.

139. In the following examples of the projection of angle-properties* the theorem to the right follows in each case from that opposite to it on the left, as appears conversely by projecting the points FF'' into the focoids.

The tangent to a circle is at right angles to the radius to its point of contact.	Any chord FF' of a conic is cut harmonically by any tangent and the line joining its point of contact to the pole C of FF'.
Confocal conics intersect at right angles.	If two conics be inscribed in a quadrilateral of which FF' are a pair of opposite summits, the tangents at any one of their common points cut FF' harmonically.
The locus of the point of concourse of two tangents to a conic which intersect at right angles is a concentric circle; or in the case of the parabola the locus is the directrix.	The locus of the point of concourse of two tangents to a conic which divide a given line FF' harmonically is a conic touching the former at FF'; or if FF' touches the original conic, the locus is the join of the points of contact of the second tangents to it from F and F'.
The locus of the intersection of tangents to a parabola which meet at a	The locus of the intersection of tangents to a conic which divide a given

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* See Salmon's *Conic Sections* §§ 356–8; Rouché et de Comberousse *Géométrie* § 1175.

given angle is a hyperbola having the same focus and directrix.	finite line FF' touching the conic in a constant cross ratio is a conic touching the former at the points of contact of the second tangents to it from F and F'.
The envelope of a chord of a conic which subtends a constant angle at one of the foci is another conic having the same focus and directrix.	If tangents SF and SF' be drawn to a conic from given points F and F', the envelope of a variable chord AB such that $S\{AFBF'\}$ is constant is a conic touching the former at its points of contact with SF and SF'.
If P be any point on a given conic, S any fixed point, and SPT an angle of constant magnitude, the envelope of TP is a conic having S for a focus.	If SFF' be fixed points, and P a variable point on a conic through and F', the envelope of a line PT such that $P\{SFTF'\}$ is constant is a conic touching SF and SF'.
If the point P be taken on a given straight line (instead of a conic), the envelope of TP becomes a parabola having S for focus.	If P be taken on a given straight line (instead of a conic), the envelope of TP becomes a conic inscribed in the triangle FSF'.

PERSPECTIVE.

140. The relation of *Perspective* in one plane may be treated either as a limiting case of the projective relation (Art. 128), or independently as follows.*

From a fixed centre of perspective S in the plane of a given figure draw radiants to all points p of the figure, and let these radiants meet a fixed axis of perspective in the same plane in points R (fig. p. 10). Then if on every radiant SR there be taken a point q such that

$$\{SpRq\} = \text{a constant},$$

the locus of q is said to be in *Perspective* with the locus of p.

Taking any two positions of SR, we have

$$\{Sp'R'q'\} = \{SpRq\};$$

and therefore pp' and qq' always intersect on the axis of perspective RR' (Art. 104). Hence also we see that to every straight line pp' in the one figure corresponds a straight line qq' in the other; and to every range $\{p\}$ in the one a homographic range $\{q\}$ in the other. Figures in perspective *in plano* are therefore homographic, and they possess the same properties as figures projectively related in space.

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* See Chasles *Sections Coniques* p. 169.

It may be shewn that if two figures in perspective in relief be turned about the line of intersection of their planes, their centre of perspective describes a circle in a plane perpendicular to that line.* [Ex. 850.

SCHOLIUM B.

The method of Projection—which is implicitly contained in the ancient theorem of Art. 103—was freely used by DESARGUES. It was used also by NEWTON, under the name *Generatio curvarum per Umbras*, in his Enumeration of Lines of the Third Order, where he remarks (p. 25, ed. Talbot): "And in the same manner as the circle, projecting its shadow, generates all the conic sections, so the five divergent parabolas, by their shadows, generate all other curves of the second genus. And thus some of the more simple curves of other genera might be found, which would form all curves of the same genus by the projection of their shadows on a plane."

Desargues also proved the fundamental property (Ex. 696) of triangles in perspective, whether in relief or *in plano* (*Œuvres* I. 413, 430). The term "homologie," for perspective in one plane, was introduced by Poncelet, and is now generally used by French writers. But since the term is in itself inexpressive, an inconvenient distinction has to be made between *homologue* and *homologique* (Rouché et De Comberousse *Géométrie* §§ 1094, 1167).

REVERSION.

141. Take fixed points S, O and a fixed straight line MN: and through the fixed points draw any two straight lines intersecting at some point R on MN, and also a pair of parallels meeting RO and RS in P and p respectively (p. 10). Then P, p may be called *Reverse Points*: O and S the *Origins of* reversion: and MN the *Base Line*. When the locus of P is a conic having S and MN for focus and directrix, we have seen that the locus of the reverse point p is the eccentric circle of O; and we have derived properties of the conic from properties of this circle.† We now proceed to treat the subject of reversion more generally. The original figure from which a reverse figure is derived may be called its *Obverse*.

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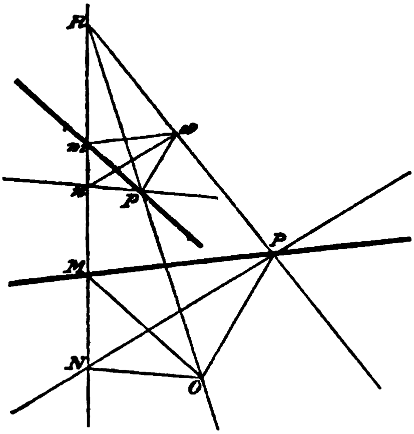
[*] Chasles *Géométrie Supérieure* §§ 368–9; Cremona *Géométrie Projective* § 90.

[†] See Arts. 4–6, 16 and Exx. 6–10.

PROPOSITION X.

142. *To any straight line drawn in a given direction corresponds a reverse line passing through a fixed point on the base line.*

Let O and ω be the origins of reversion: P and p any two reverse points: ωm and PM a pair of parallels, meeting the base line in M and m.



Then, if R be the point on the base line at which $P\omega$ and Op intersect,

$$OP : \omega p = PR : \omega R = PM : \omega m,$$

and therefore OM and pm are parallel.

Hence, if ωm be a fixed line and P a variable point on any assumed line parallel to ωm, the locus of p is the straight line drawn through the fixed point m on the base line parallel to OM.

Corollary 1.

The point at infinity on any system of parallels PM corresponds to a reverse point m on the base line. All points at infinity in the same plane are therefore to be regarded as lying in one straight line, of which the base line is the reverse.

Furthermore *the direction of the line at infinity is indeterminate.* For, as pm turns about the same point m on the base line, the reverse line PM remains parallel to ωm: and ultimately, when pm coalesces with the base line, PM becomes the *line at infinity*, which may accordingly be regarded as parallel to any assumed line ωm.

Corollary 2.

If the arms of any angle MPN and of the reverse angle mpn meet the base line in M, N and m, n respectively, then

$$\angle m\omega n = MPN; \text{ and } \angle MON = mpn.$$

Notice that to every angle PSQ subtended *at either origin* S (fig. Art. 4) corresponds an *equal* reverse angle pOq subtended at the other. For example, the angles PSO, pOS are equal, in the figure of Art. 6.

PROPOSITION XI.

143. *Any straight line being taken as base line, any two given angles may be reversed into angles of given magnitudes.*

For the angle MPN (Prop. X. Cor. 2) reverses into an angle of given magnitude α, if the origin O be taken on the circular segment MON described on MN so as to contain an angle equal to α. By a like construction a second angle may be reversed into an angle of given magnitude β. And if O be taken at the intersection of the two segments, the two angles will reverse simultaneously into angles equal α and β respectively.

The applications of this general theorem are precisely analogous to those of the corresponding theorem in Conical Projection. [Prop. V.

Corollary.

From any origin O a given conic may, by properly choosing the base line, be reversed into a conic through two given points at infinity, whose magnitude is then determined by the position of the reverse origin ω. Or if the base line be given, the origin O may be determined by reversing the angles between two assumed pairs of lines PA, PB and PC, PD—which may be drawn conjugate with respect to the given conic—into angles α and β respectively. By properly determining the origins and the base line together we may reverse *any conic and point U and P into any conic and point U' and P'*. For example, if α and β be right angles and the polar of P with respect to U be taken as base line, U' becomes a circle whose centre is P'. [Prop. VIII.

144. The following are some applications of the property of reverse figures that *all angles subtended at the origin in the one figure correspond to equal angles subtended at the reverse origin in the other.* [Art. 142 Cor. 2.

a. A variable chord of a conic which subtends a right angle at a given point envelopes a conic having that point for a focus.

For if the given point O and its polar be taken as origin and base line, the reverse conic has its centre at the reverse origin ω (Art. 142 Cor. 1); and a variable chord of the latter conic which subtends a right angle at ω envelopes a concentric circle (Ex. 289), of which the obverse is a conic having O and the base-line for a focus and directrix.

b. A variable chord of a conic which subtends a right angle at a given point on the curve passes through a fixed point on the normal thereat.[*]

For if a conic through O be reversed into a circle through ω, every chord of the former which subtends a right angle at O has for its reverse a diameter of the circle, and therefore passes through the fixed point which is the obverse of the centre of the circle. Note that the tangents and also the normals at O and ω are reverse lines.

Hence, to reverse a conic from any point O upon it as origin into a circle, we must have as base line *the polar BC of the point of concourse of all chords which subtend right angles at O.*

c. Let DOE be a fixed angle inscribed in a conic, P any point on the curve, B and C the points in which PD and PE meet the polar of the point of concourse of all chords which subtend right angles at O: *then will the angle BOC be equal or supplementary to the angle DOE.*[*]

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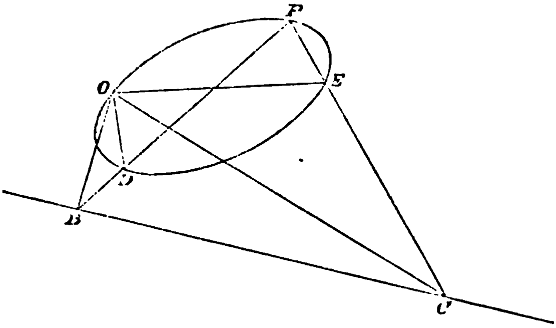
[*] This theorem of Frégier—p. 276, note, and *Correspondance sur l'Ecole Royale Polytechnique* tome III p. 394, 1816—is a limiting case of § *a*. See Scholium D, p. 285.

For if with O as origin and BC as base line the conic be reversed into a circle, then (with the same notation) the points B and C are removed to infinity, and the theorem follows at once from the equality of the angles DOE and DPE in the same segment of the circle.



d. If a straight line PDB turning about a fixed point P meet the arms of a constant angle BOD, which turns about a fixed point O, in B and D; then if the point B moves along a straight line BC, *the point D describes a conic through O and P.* For when BC is the line at infinity the locus of D is evidently a circle through O and P; and therefore by reversion, the locus of D in the general case is a conic through O and P.

This is a limiting form of Newton's *Descriptio Organica* (Art. 113 Cor. 1), since the line through P may be regarded as a vanishing angle BPD.

e. Every range $\{ABCD\}$ and its reverse $\{abcd\}$ subtend similar pencils $O\{ABCD\}$ and $\omega\{abcd\}$ at the origins, and are therefore homographic. All the properties of cross ratio may therefore be extended from the circle to the general conic by reversion.

145. *The Orthocentre.*

a. Let the sides of a triangle ABC, the reverse of $A'B'C'$,

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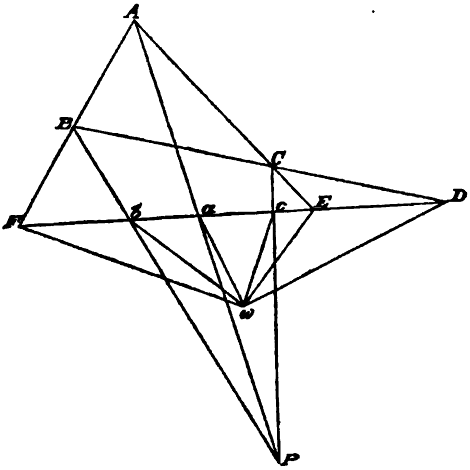
[*] See *Mathematical Questions from the* EDUCATIONAL TIMES, vol. I. pp. 33, 40 (Question 1409).

meet the base line in points D, E, F, and let aD, bE, cF be segments of the base line which subtend right angles at ω. Then Aa, Bb, Cc are reverse to the perpendiculars of the triangle $A'B'C'$ (Art. 142 Cor 2) and cointersect at the reverse P of its orthocentre.

It is hence evident that the sides of any triangle ABC and the radiants from any point to its vertices determine an involution $\{aD, bE, cF\}$ on any transversal. [Art. 110.



b. If the triangle ABC envelopes a fixed conic touching the base line, the obverse of which is a *parabola*, the point P traces a straight line, the reverse of the directrix of the parabola. [Art 29 Cor. 1.

Or if, starting with the parabola and taking its directrix as base line, we reverse it into a circle about ω as centre, the point P is removed to infinity. Hence, if the sides of a triangle ABC touch a circle, and meet any fourth tangent to it in abc, and if the diameters parallel to the polars of abc meet the fourth tangent in DEF, the lines AD, BE, CF are parallel. In other words:

If the sides of a triangle ABC touch a circle, and if the parallel tangents meet any seventh tangent in DEF, the lines AD, BE, CF are parallel.

More generally:

If from three collinear points XYZ pairs of tangents be drawn to a conic, and if ABC be the triangle formed by one tangent from each pair and DEF the points in which the remaining three tangents meet any seventh tangent, the lines AD, BE, CF meet at a point in a straight line with XYZ.

146. *The Normal.*

The reverse of the normal at any point P to a conic is the line through the reverse point p which with the tangent at p intercepts on the base line a length which subtends a right angle at the reverse origin ω. [Art. 142 Cor. 2.

147. *Conjugate Diameters.*

If a conic be reversed into the eccentric circle of ω, it may be seen that a pair of its conjugate diameters inclined at angles α and $\pi-\alpha$, reverse into lines through the pole of the base line with respect to the circle and which contain angles $\pi-\alpha$ and α.

148. *The Asymptotes.*

If a conic meets the base line in M and N, the asymptotes of its reverse correspond to the tangents at M and N, and are therefore parallel to MO and NO, where O is the origin (Art. 142). We may therefore determine the eccentricity of the reverse conic by making the angle MON of any assumed magnitude, real or imaginary.

SCHOLIUM C.

REVERSE lines OP and $p\omega$ through the origins (which may be supposed to lie on the same side of the base line) being reverse in direction, figures are consequently, in a manner, *turned over* in this transformation, so that an original figure and its derivative may be regarded as obverse and reverse respectively. Thus in Art. 4, if the circle be divided by axes through O parallel and at right angles to the base line, its first and third quadrants must be turned over or interchanged, and likewise its second and fourth, in order that they may become similarly situated with the sectors of the conic to which they severally correspond.

If reverse points P and p be referred to rectangular axes of coordinates, the base line being the common axis of x and the axes of y being drawn through O and ω respectively, then if X, Y be the coordinates of P and x, y those of p, and if OD and ωd be the ordinates of O and ω it may be shewn that $Yy = \omega d . OD$, and $X : -x = Y : \omega d$. Hence an equation of any degree between x and y implies an equation of the same degree between X and Y.

Reversion is a special case of the following transformation. Take fixed origins O and ω, and a fixed director line (or plane) corresponding to each: from any point P draw POD to meet the O-director and $P\omega d$ to meet the ω-director: then the point of concourse of ωD and Od corresponds to P. The construction in the text results from supposing one of the directors to be at infinity. The analysis for the general case is fully given in a section by Prof. Cayley contributed to my article on the Homographic Transformation of Angles in the *Quarterly Journal of Mathematics* XIV. 25—39.

HOMOGRAPHIC TRANSFORMATION.

PROPOSITION XII.

149. *Any two plane homographic figures of the same species are capable of being placed in perspective.*

We have seen that any two plane figures in perspective are so related that to every range in the one corresponds a homographic range in the other (Prop. IV). Conversely, any two plane figures thus related are capable of being placed in perspective.

(i) For if $ABCD$ be four fixed points and P a variable point in a plane figure, and $A'B'C'D'P'$ the corresponding points in a homographic figure, it is evident from the relation,

$$P\{ABCD\} = P'\{A'B'C'D'\},$$

that by projecting the points $ABCD$ into $A'B'C'D'$ (Art. 135) we at the same time project every point P into its correspondent P'.

(ii) The same result may also be arrived at as follows.

Let A be a given plane figure, regarded as moveable in any way in its plane, and B a fixed homographic figure in the same plane. Then to the focoids ϕ and ϕ', regarded as belonging to A, correspond fixed points F and F' related to B.

[Art. 132.

Let $F\phi$ and $F'\phi'$ meet in the point O related to B, and let O' be the corresponding point in A. Also let P and Q be any two points in B, and P' and Q' the corresponding points in A.

Move the figure A a certain distance in a certain direction until O' coincides with O, and then turn it about O until the points POP' are brought into one straight line.* Then, since A and B are homographic,

$$O\{PQFF'\} = O\{P'Q'FF'\};$$

and therefore, since three rays in the one pencil coalesce severally with three in the other, the fourth ray OQ coalesces with the fourth OQ', or every two corresponding points Q, Q' are in a straight line with O, the required centre of perspective of A and B. It then follows from Art. 140 that A and B may be placed perspective in space.

Corollary.

Since figures homographic with the same figure are homographic with one another, and since any conic and an assumed point in its plane may be projected into a circle and its centre (Art. 137), and conversely; it follows that *any conic and point in one plane may be projected into any other conic and point in one plane.* [Prop. XI. Cor.

SCHOLIUM D.

Transformation is a convenient (if not strictly accurate) expression for the derivation of one figure from another in accordance with an assigned law of correspondence. The general idea of homographic transformation may be found in a passing remark of DESARGUES (*Œuvres* I. 214), who, having enunciated the fundamental property of the polar planes of a sphere, concludes by stating curtly that it may be extended to surfaces which are related to the sphere as the ellipse is to the circle: "Semblable propriété se trouve à l'egard d'autres massifs qui ont du rapport à la boule, comme les ouales autrement ellipses en ont au cercle, mais il y a trop à dire pour n'en rien laisser."

In the tract on *Plani-coniques* appended to his *Nouvelle méthode en Géométrie &c.* (Paris 1673), De la Hire derived the general conic from the circle by a geometrical transformation *in plano.* A similar transformation appears to have been arrived at by Le Poivre about thirty years later (Chasles *Sections Coniques* p. 174). NEWTON shewed how to transform curves of all orders (*Principia* Lib. I. sect. V. lemma 22) by substitutions of the form $X=\frac{ab}{x}$ and $Y=\frac{ay}{x}$ (cf. Schol. C), and added two examples of the application of his method (*loc. cit.* props. 25, 26). The "collinear" figures of Möbius "sont aussi des figures homographiques les plus générales" (Chasles *Sections Coniques* p. 165). Möbius proved *inter alia* that any four points in a plane may be projected into any other four points in a plane (*Barycentrische Calcul* p. 327, 1827). For a general exposition of the principle of Homography see Chasles' *Mémoire* on duality and homography, at the end of his *Aperçu historique.* See also his *Géométrie Supérieure* pp. 362—412, and Townsend's *Modern Geometry* chaps. 19—22.

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* See Salmon's *Higher Plane Curves* Art. 330.

EXAMPLES.

[*It is left to the reader in some cases to modify the enunciations of the pairs of theorems in the double columns so as to bring them into exact correspondence*].

801. The diagonals of a parallelogram and the lines bisecting its two pairs of opposite sides form a harmonic pencil.	Each diagonal of a complete quadrilateral is divided harmonically by the remaining two diagonals.
802. If two triangles be similar and similarly placed, the joins of their corresponding vertices meet in a point.	If the intersections of the three pairs of sides of two triangles lie in one straight line, the joins of the opposite vertices cointersect.
803. Any two pairs of parallels through points P and Q meet any transversal in an involution having its centre on PQ.	The three pairs of joins of any four points in a plane determine an involution on any transversal.
804. The centres of the diagonals of a complete quadrilateral are in one straight line. [p. 256.	An infinity of pairs of straight lines can be found which divide the three diagonals of a quadrilateral harmonically.
805. Parallel chords of a circle are bisected by a straight line through its centre.	Concurrent chords of a conic are divided harmonically by their common point and its polar.
806. If two of the three pairs of opposite sides of a hexagon inscribed in a circle are parallel, the third pair are parallel.[*]	The three pairs of opposite sides of any hexagon inscribed in a conic have their intersections in one straight line.
807. A system of coaxal circles meet any transversal in pairs of points in an involution. [Art. 109.	All the conics through four given points meet any transversal in pairs of points in an involution.

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[*] See Gergonne's *Annales* IV. 79.

808. Four circles can be drawn touching three given straight lines.	Four conics can be drawn through two given points and touching three given lines.
809. Four conics can be drawn through three given points so as to have a given point for a focus.	Four conics can be drawn through three given points so as to touch two given lines.
810. The diameters of a circle subtend a pencil in involution at any point on the circumference.	A system of concurrent chords of a conic subtend a pencil in involution at any point on the curve. [p. 276.
811. Given two pairs of lines conjugate with respect to a circle, the locus of its centre is the rectangular hyperbola circumscribing the quadrilateral of which the conjugate lines are opposite sides.[*]	Given a chord FF' of a conic and two pairs of lines conjugate with respect to it, the locus of the pole of FF' is a conic with respect to which F and F' are a pair of conjugate points.
812. Given three pairs of lines conjugate with respect to a circle, the positions of its centre constitute an orthocentric tetrastigm.	Through two given points four conics can be drawn so as to have three given pairs of lines conjugate with respect to them; and their common chord is divided harmonically by every conic through its four poles with respect to them.
813. Every circle through the centre of a rectangular hyperbola circumscribes an infinity of triangles self-polar with respect to the hyperbola.	If two triangles be self-polar with respect to a conic, their six angular points lie on a conic.
814. If a triangle PQR right angled at P be inscribed in a rectangular hyperbola, the perpendicular from P to QR is the tangent at P.	If F and F' be conjugate points with respect to a conic, PQ and PR any two chords which divide FF' harmonically; then QR and the tangent at P divide FF' harmonically.
815. The directions of two sides of a triangle inscribed in a circle being given, the envelope of the third side is a concentric circle.	If two sides of a triangle inscribed in a conic pass each through a given point, the envelope of the third side is a conic touching the former at two points on the join of the given points.
816. The envelope of the polar of any point on a circle with respect to a concentric circle is a concentric circle.	The envelope of the polar of any point on a conic with respect to a second having the same focus and directrix is a third having the same focus and directrix.

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[*] From the centre O of the circle draw a perpendicular OP to one of the lines, and let it meet the conjugate line in Q; and draw OP' perpendicular to one of the second pair of lines, and let it meet the fourth line in Q'. Then since $OP \, . \, OQ = (\text{radius})^2 = OP' . \, OQ'$, the locus of O is a conic through the four vertices of the quadrilateral; and it is easily seen that the orthocentre of any three of them is a point on the locus.

817. Parallel chords of a circle are cut in a constant ratio by a concentric ellipse touching the circle at the extremities of the perpendicular diameter.	Concurrent chords of a conic are divided in a constant cross ratio by every conic having double contact with the former upon the polar of the point of concurrence.
818. Every parallelogram inscribed in a circle is rectangular.	The intersections of the two diagonals and of the opposite sides of any quadrilateral are a conjugate triad with respect to every conic circumscribing the quadrilateral.
819. The diagonals of every parallelogram circumscribed to a circle meet at right angles at its centre.	The diagonals of a complete quadrilateral are a conjugate triad with respect to every conic inscribed in it.
820. The centres of all the rectangular hyperbolas circumscribed to a given triangle lie on its nine-point circle.	Given four points on a conic, the locus of the pole of a given line is a conic, &c. [Ex. 788.
821. The circumscribed circle of every triangle which circumscribes a parabola passes through its focus.	If two triangles circumscribe a conic, their six summits lie on a conic.
822. The envelope of the polar of a given point with respect to a system of confocal conics is a parabola touching their axes and having the given point* for a point on its directrix. [Ex. 379.	The envelope of the polar of a given point with respect to the system of conics inscribed in a quadrilateral is a conic touching its three diagonals; and the chord of contact of the second tangents to this conic from the extremities of any diagonal of the quadrilateral is the line joining the given point to the point of concourse of the remaining two diagonals.
823. If from a fixed point O tangents OP and OQ be drawn to any one of a system of confocal conics, the circle through OPQ passes through a second fixed point. [Exx. 340, 380.	If from a fixed point O tangents OP and OQ be drawn to any one of a system of confocal conics, the conic through their foci and OPQ passes through a fourth fixed point.†
824. Given three concentric circles, any tangent to one of them is divided into segments of constant lengths by the remaining two.	If three conics touch one another at the same two points, any tangent to one of them is divided in a constant cross ratio by the remaining two.
825. Four fixed tangents to a parabola divide any fifth tangent into segments whose ratios are constant. [Ex. 726.	Four fixed tangents to a conic divide any fifth tangent in a constant cross ratio.

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* The tangents at this point to the two confocals through it touch the parabola.

† Project the common foci of the first system into the focoids of the plane of the second. See the *Quarterly Journal of Mathematics* x. 287.

826. Any two pairs of points which divide the diagonals of a rectangle harmonically lie on a circle.

If the extremities of each of two diagonals of a quadrilateral are conjugate points with respect to a conic, the extremities of the third are conjugate with respect to it.[*]

827. If POp, QOq, ROr, SOs be any four concurrent chords of a conic, the conics through $OPQRS$ and $Opqrs$ have a common tangent at O.

828. If FF' be a common chord of two given conics, its pole with respect to any conic which touches both of them and passes through F and F' has for its locus a conic touching the tangents at F and F' to the given conics.

829. If a conic touches the sides SF and SF' of a given triangle and also two other given lines, the second tangents to it from F and F' meet on a fixed straight line.

830. Given, in addition to a chord of a conic, two tangents, or one tangent and one point, find in each case the locus of the pole of the given chord.

831. If two conics have double contact, the cross ratio of four of the points in which any four tangents to the one meet the other is equal to that of the remaining four points, and also to that of the points of contact taken in the same sense of rotation.[†]

832. Extend by projection Newton's theorem, that the diameter of a quadrilateral is the centre-locus of all conics inscribed therein.[‡] [p. 282.

833. The circle through any triad of points conjugate with respect to a conic is orthogonal to its orthocycle (pp. 274, 280). Is this theorem projective?

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[*] This theorem and its reciprocal are due to HESSE (Crelle's *Journal* XX. 301, 1840).

[†] See Salmon's *Conic Sections* Arts. 276, 354.

[‡] Its analogue in space was given in Gergonne's *Annales* XVII. 200.

834. From a fixed point O tangents OP and OQ are drawn to any conic inscribed in a given quadrilateral, and through P and Q are drawn the straight lines which with PO and QO respectively divide the three diagonals of the quadrilateral harmonically. Shew that the lines so drawn touch the fixed conic which is the envelope of PQ. [Ex. 822.

835. If AB be one of the diagonals of this quadrilateral, the conic through $ABOPQ$ passes through a fourth fixed point O', such that AO, AO' and BO, BO' divide the remaining two diagonals harmonically. Shew also that the three positions of O' corresponding to the three diagonals of the quadrilateral lie in one straight line.

836. If from a fixed point O tangents OP and OQ be drawn to any one of a system of confocal conics, and if the normals at P and Q meet in N; the locus of the orthocentre of the triangle NPQ is a straight line, and the locus of the orthocentre of OPQ is a rectangular hyperbola having one asymptote parallel to the central distance of O.* What do these theorems become by projection?

837. Given the orthocentre of a triangle inscribed (or circumscribed) to a given conic, the product of the segments of its perpendiculars is constant. Hence shew that if one solid angle contained by three planes mutually at right angles can be inscribed (or circumscribed) to the surface of a given cone of the second degree, an infinity of such angles can be inscribed (or circumscribed) to it.

838. The locus of the point in space from which triads of lines mutually at right angles can be drawn to triads of points on a given conic is a sphere.

839. If the conic be supposed to vary, yet so as always touch the sides of a given quadrilateral, the sphere will pass

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* See the *Quarterly Journal of Mathematics* x. 289.

through a fixed circle in a plane at right angles to the plane of the quadrilateral.[*]

840. ABC is a triangle inscribed in a conic: Aa, Bb, Cc are chords drawn through a point O: and Ab, Bc, Ca meet the polar of O in PQR respectively. Shew that the lines from any point on the conic to the points PQR respectively meet the sides of ABC in three collinear points.

841. Rays being drawn from a fixed point on a conic, shew that the intercepts upon them between the conic and a fixed tangent may all be projected upon a given line through the fixed point, from another point on the conic, into segments of the same length.

842. If a point P on a conic be connected with two fixed points F and F' in its plane, all the chords which are divided harmonically by FP and $F'P$ are concurrent; and the locus of their point of concourse, as P varies, is a conic touching the first at two points on FF'.

843. If a conic passes through two given points and touches a given conic at a given point, its chord of intersection with the given conic passes through a fixed point.

844. $ABCD$ being four points on a conic, E and F are the poles and O is the point of concourse of AB and CD. Through E is drawn a straight line meeting CD in M and the conic in Q and R; and upon this line a point P is taken a fourth harmonic to QMR. Shew that the locus of P is the conic through $ABCDEF$, the tangents to which at E and F pass through O; and that the tangents from O to the first conic pass through the four points in which the common tangents to the two conics touch the second.

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[*] On Exx. 837–9 see Picquet's *Etude géométrique des Systèmes Ponctuels et Tangentiels de Sections Coniques* §§ 72, 73, 85, 86. Picquet now uses the term *orthoptic circle* (p. 42) to denote the orthocycle, and the term *orthoptic summits* (p. 41) of the "pencil" of conics inscribed in a quadrilateral to denote the two fixed points on their orthocycles. Gaskin's discovery of these points was anticipated by Plücker (*Analytisch-geometrische Entwicklungen* II. 198, 1831).

845. If a variable conic has double contact with each of two fixed conics, find the loci of the points of concourse of its common tangents with the fixed conics.

846. If aa', bb', cc' be parallel chords of a conic and p any seventh point on the curve, prove that the three points $(ap, b'c')$, $(bp, c'a')$, $(cp, a'b')$ lie on a straight line parallel to the chords.

847. If from each of four points on a circle perpendiculars be drawn to the joins of the remaining three, the feet of these perpendiculars lie by threes on four concurrent lines. Generalise this theorem by reversion, or otherwise.

848. Any two conics may be regarded as homographic figures in which any three points on the one correspond to three points taken arbitrarily on the other.*

849. Shew how to place reverse figures in perspective, and adapt the constructions of Arts. 141–148 to the case of figures in perspective in one plane.

850. The construction in Art. 134 (ii) for fixing the position of V in the vertex-plane is independent of the angle between that plane and the primitive plane: as this angle varies the vertex remains fixed in its plane: it therefore describes a circle in a plane perpendicular to ab, or to the intersection of the plane of projection with the primitive plane.

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* Chasles *Sections Coniques* p. 167.

NOTE.

The undermentioned Examples (cf. p. 141) are taken from the EDUCATIONAL TIMES *Reprint* :

Ex. 441 (i. 52); 442 (xi. 31); 443 (v. 101); 444 (vii. 49); 445 (xxx. 90); 450 (xii. 27); 454 (viii. 72); 455 (iii. 35); 466 (xxx. 90); 516 (i. 55); 517 (v 56); 518 (i. 53); 520 (v. 36); 521 (vii. 96); 522 (xi. 59); 523 (xviii. 32); 524 (xviii. 54); 526 (xxi. 35); 527 (xxiv 87); 560 (xvii. 35); 562 (xviii. 52); 563 (xviii. 98); 578 (vi. 71); 607 (xv. 88); 611 (xxv. 23); 612 (xxv. 21); 613 (iv. 70); 620 (xxiv. 43); 638 (iv. 97); 640 (xiii 60); 644 (v. 87); 645 (xi. 104); 680 (iv. 69); 681 (x. 32); 682 (xvii, 60); 684 (xxii. 22); 685 (xxii. 51); 687 (xxiv. 101); 688 (xxvi. 69); 689 (xxviii. 95); 690 (xxxi. 18); 714 (xi. 60); 727 (xiii. 26); 737 (xxv. 61); 738 (ii. 6); 835 xiii. 61); 840 (iii. 39); 841 (ix. 79); 842 (xi. 100); 843 (xii. 54); 844 (xiv. 80); 845 (xix. 101); 846 (xxiii. 94); 847 (xxx. 64. xxviii. 57).

CHAPTER XII.

RECIPROCATION AND INVERSION.

150. WE now return to the method of Reciprocal Polars, of which some account was given in Chapter X. [p. 268.

Take any fixed conic as *Director*, and let any straight line or point be said to *correspond* or be *reciprocal* to its pole or polar with respect to the director. It is evident that to the intersection of any two lines corresponds the join of the reciprocal points, and that to a system of concurrent lines correspond a system of collinear points. [Art. 17 Cor. 1.

To any curve, regarded as the envelope of its tangents, corresponds the locus of their poles with respect to the director; and to the join of any two *consecutive* points on either curve corresponds the join of two consecutive tangents to the other. Hence, to every tangent and its point of contact in either figure correspond a point and the tangent thereat in the other.

It is hence evident that if two curves touch one another in one or more points, their reciprocals touch one another in the same number of points.

Notice that *the tangents to the director at its intersections with any curve are also tangents to the reciprocal curve.*

PROPOSITION I.

151. *The degree of any curve not having singular points is equal to the class of its reciprocal with respect to a conic, and conversely.*

(i) For if U be any curve and u its reciprocal, A any straight line and a its reciprocal; then to every point in which A meets U corresponds a tangent to u, and every such tangent passes through a. The number of points in which an arbitrary line A meets U is therefore equal to the number of tangents that can be drawn from any one point to u, as was to be proved.

It follows that the reciprocal of a *conic* is a conic, as was otherwise shewn in Art. 117; and that the reciprocals of all conics touching the same four lines, or *having the same foci*, are conics passing through the same four points. [Prop. VI.

PROPOSITION II.

152. *To every point and its polar with respect to a conic correspond a straight line and its pole with respect to the reciprocal conic: to every conjugate triad of points in the one figure a conjugate triad of lines in the other: and to every pencil or range in the one a homographic range or pencil in the other.*

(i) For if ab and ac be the tangents from any point a to a conic, then to their points of contact b and c correspond a pair of tangents AB and AC to the reciprocal conic; and to the points of contact B and C correspond reciprocally the tangents at b and c to the original conic. [Art. 150.

It follows that the join of B, C corresponds to a, and the join of b, c to the point of concourse A of the tangents at B and C.

Hence to any point a and its polar bc in the one figure correspond a line BC and its pole A in the other; and therefore to every conjugate triad of points in either corresponds a conjugate triad of lines in the other.

(ii) It has been shewn in Art. 116 that every row of points and their polars with respect to any director are homographic.

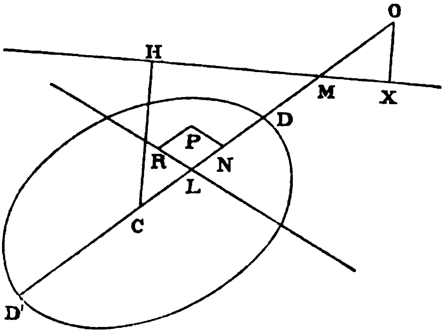
Corollary.

Since the points of concourse of a conic and its reciprocal correspond to their common tangents, *the cross ratio of the four common points of any two conics in either is equal to that of their common tangents in the other.* [Ex. 727.

PROPOSITION III.

153. *Given in a plane two straight lines and their poles with respect to a conic, the ratio of the distances of any point in the plane from the given lines varies as the ratio of the distances of its polar from their poles.*[*]

(i) Let the polars of any two points O and P with respect to a conic whose centre is C meet CO in L and M respectively, and let PN be an ordinate to the diameter CO.



Then since $CL.CO = CM.CN$,[†]

or $CM + OM : CM = CL + NL : CL$;

therefore $OM : CM = NL : CL = PR : CL$,

if PR be drawn parallel to CL to meet the polar of O. That is to say, *the distances of O and C from the polar of P are as the distances of P and C from the polar of O.* [Ex. 325.

(ii) If OX and CH be perpendiculars to the polar of P, it follows that

$$PR : CL = OX : CH;$$

and the same proportion will hold if PR and CL be now supposed *perpendicular* to the polar of O.

In like manner, taking any second position o of O whilst P remains as before, we have

$$Pr : Cl = ox : CH.$$

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[*] Chasles *Aperçu historique* p. 590, 1875.

[†] If CM meets the conic in D and D', and if ∞ be the polar of DD', the point M and its polar $P\infty$ divide DD' harmonically. Hence $CM.CN = CD^2$, whatever be the position of P. This follows also by orthogonal projection from Ex. 281.

Hence, whatever be the position of P, the ratio $\frac{PR}{Pr}$ is equal to $\frac{CL}{Cl} \cdot \frac{OX}{ox}$, where CL and Cl are constant for given positions of O and o; that is to say, *the ratio of the perpendiculars from a variable point P to the polars of two fixed points O and o varies as the ratio of the perpendiculars from O and o to the polar of P.* We may of course, as a special case, suppose either point and its polar to become a point on the curve and the *tangent* thereat.

Corollary.

It is hence evident that any homogeneous relation between the distances of a variable point P from any number of fixed straight lines implies a homogeneous relation of the same degree between the distances of the polar of P with respect to a conic from the poles of the fixed lines. Thus from the *Locus ad quatuor lineas*, $PQ.PR = k.PS.PT$ (Scholium C, p. 266), we deduce a relation of the same form between the distances of the tangent at P from two pairs of fixed points, opposite vertices of a quadrilateral circumscribed to the conic. [Ex. 707.

POINT RECIPROCATION.

154. If O be the centre of a circle, OA the perpendicular from it to any straight line L and a the point on OA or its prolongation such that $OA.Oa = (\text{radius})^2$; then a is the pole or reciprocal of L with respect to the circle. The same point a may also be determined by regarding O merely as a fixed point, without reference to the circle, and taking $OA.Oa$ equal to a constant quantity c^2. This last construction is called *reciprocation with respect to a point*, the point being called the *origin*. When c^2 is *negative* every line or point and its reciprocal lie on opposite sides of the origin. The construction is then equivalent to reciprocating with respect to an *imaginary* circle. [Prop. v. (ii).

Notice that to *the foot of the perpendicular A* from the origin to any straight line L corresponds the line through the reciprocal point a parallel to L or at right angles to Oa;

and that to each line in *a system of parallels* corresponds a point on the common perpendicular to them from the origin.

PROPOSITION IV.

155. *Any two straight lines contain an angle equal to that subtended by their point-reciprocals at the origin; and the distance of any straight line from the origin varies inversely as the distance of its reciprocal therefrom.*

The first part of the proposition follows immediately from the perpendicularity of every straight line to the line joining its reciprocal to the origin.

The second part is merely another way of stating that the product $OA . Oa$, in the construction of Art. 154, is equal to a constant c^2.

Corollary 1.

From the relation $Oa = \frac{c^2}{OA}$ we deduce, that to any straight line L through O corresponds the point *at infinity* in the direction at right angles to L, and conversely. All points at infinity in the same plane therefore lie on the reciprocal of the origin, and are consequently to be regarded as collinear. To the polar of the origin with respect to the original conic corresponds *the centre of the reciprocal conic*, the polar of the line at infinity with respect to it. [Art. 17 Cor. 2.

Corollary 2.

To the tangents OP and OP' from the origin to any conic correspond the points at infinity in the directions at right angles to OP and OP'; the *eccentricity* of the reciprocal conic is therefore determined by the angle POP', the supplement of the angle between its asymptotes. Hence the reciprocal will be a hyperbola, an ellipse or a parabola according as O is taken without, within or upon the original conic. Thus we see again that *every parabola touches the line at infinity*, the reciprocal of the origin. The reciprocal of a conic with respect to any point at which it subtends a right angle is a *rectangular hyperbola.* In any case the *axis* of the reciprocal conic is parallel to the bisector of the angle POP'.

Corollary 3.

From either point of concourse of their orthocycles as origin any two conics U and V reciprocate into rectangular hyperbolas u and v; and conversely every conic through the four intersections of u and v, being itself a rectangular hyperbola (Art. 69), is reciprocal to a conic subtending a right angle at the origin and touching the four common tangents to U and V. Hence *the orthocycles of all the conics which touch four given lines have two points in common.* [Art. 123 Cor. 1.

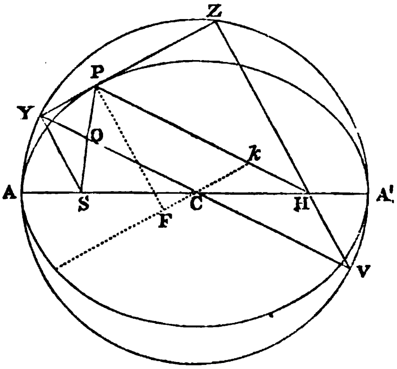
PROPOSITION V.

156. *The reciprocal of a circle with respect to any point is a conic having that point for a focus, and conversely.*

(i). If the director be a circle about the origin O as centre, any other circle C meets it at the focoids ϕ and ϕ', and therefore has for its reciprocal a conic touching $O\phi$ and $O\phi'$, the tangents to the director at the focoids. [Art. 130.

This conic therefore has O for a focus; and *its O-directrix* (the polar of O) corresponds to the pole of the line at infinity with respect to C, that is to say, it *corresponds to the centre of C.*

(ii). Otherwise thus. Reciprocate a conic from either focus H as origin, and let V be the point corresponding to any tangent to the conic, and Z the projection of H upon that tangent.



Then if CB be the conjugate semiaxis, and $-CB^2$ be taken as the constant of reciprocation,

$$HV.HZ = -CB^2,$$

or V lies on the major auxiliary circle, which is accordingly a reciprocal of the conic with respect to either focus. Hence, whatever be the constant of reciprocation, the reciprocal of a conic with respect to either focus is a circle, and conversely.*

In the limiting case of the *parabola*, produce SY and SA (fig. Art. 26) to Z and B, so that $SY.SZ = SA.SB =$ a constant. Then Z lies on the circle upon SB as diameter, which is accordingly a reciprocal of the parabola with respect to S, and conversely.

(iii). In the diagram, the *eccentricity* of the conic is equal to $\frac{CH}{CA}$, and its *latus rectum* to $\frac{2CB^2}{CA}$.

Moreover, if X be the foot of the H-directrix,

$$HC.HX = -CB^2, \qquad \text{[Art. 35 Cor. 3}$$

and therefore (1) the H-directrix is reciprocal to the centre of the *circle*, and (2) the polar of H with respect to the circle—which in this case coincides with the H-directrix (Art. 35 Cor. 1)—is reciprocal to the centre of the *conic*.

If the constant of reciprocation be changed the relative *magnitude* of the conic and the circle will alter, but the following relations will still be found to subsist. The origin is now denoted by O, and the centre of the *circle*—which will in general be distinct from that of the conic—by C.

$$\text{eccentricity} = \frac{\text{distance of } C \text{ from } O}{\text{radius of circle}}.$$

$$\tfrac{1}{2}\text{ latus rectum} = \frac{\text{constant of reciprocation}}{\text{radius of circle}}.$$

centre of circle $\equiv$ reciprocal of O-directrix.

centre of conic $\equiv$ reciprocal of polar of O with respect to circle.

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* Another proof has been given by Laquière, *Nouvelles Annales* xx. 42, 1861; and another by Salmon, *Conic Sections* Art. 308.

Corollary 1.

One inscribed and three escribed circles can be drawn to a given triangle, and their radii are connected by the relation,

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}.$$

Hence, by reciprocation with respect to any point O, four conics can be described having a given point for a focus and passing through three given points, and the latera recta of three of them are together equal to the latus rectum of the fourth.

Corollary 2.

The point V in the diagram corresponds to the tangent YZ; the point C to the H-directrix; the line VC to the point of concourse D of YZ with the H-directrix; the point of concourse Y of VC and YZ to the line DV. But Y lies on the circle; therefore DV touches the conic. The second point V' in which DV meets the circle corresponds to the second tangent from Y to the conic. This tangent is evidently parallel to the opposite tangent DV; therefore YHV' is at right angles to DV and H is the orthocentre of the triangle DYV. [Ex. 330.

PROPOSITION VI.

157. *All the circles of a coaxal system reciprocate from either of their limiting points into confocal conics.*

(i). Whatever be the position of the origin O in the plane of the circles, their reciprocals have O for a common focus.

If the origin be taken at either limiting point* of the system of circles, it has the same polar with respect to them all, and therefore the line at infinity has the same pole with respect to all their reciprocals. That is to say, the latter have a common *centre* as well as one focus in common. They are therefore confocal, as was to be proved.

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* These points are the limits of the system "par rapport à l'*infiniment petit*," and the radical axis and the line at infinity are its limits "par rapport à l'*infiniment grand.*" When a circle becomes infinite it degenerates in general into a straight line at a finite distance *together with the line at infinity*. See Poncelet's *Traité des Propr. Projectives* p. 49 (1822); Townsend's *Modern Geometry* I. 199.

(ii). The system of conics through four given points reciprocate with respect to any one of the system as director into conics touching the tangents to the director at the given points. The construction § i. corresponds to the case in which the focoids are two of the four given points, and the director reduces to a point-circle.

Corollary.[*]

The further limiting point O' of the circles corresponds to the *minor axis* of the confocals, and the line bisecting OO' at right angles to their *further focus*. The orthogonal circles through O and O' reciprocate at the same time into *parabolas touching the minor axis of the confocals, and whose directrices pass through the further focus of the confocals;* and every common tangent to one of the confocals and one of the parabolas subtends a right angle at O.

SCHOLIUM A.

Reciprocation presupposes the idea of an envelope, which originated, according to Montucla (*Hist. des Mathématiques* tome II. 120, 1758), with Florimond de Beaune (1601—1651), a zealous advocate of the new Cartesian geometry. In a letter to De Beaune dated 20 fev. 1639, Descartes writes: "Pour vos lignes courbes, la propriété dont vous m'envoyez la démonstration me paroît si belle que je la préfère à la quadrature de la parabole trouvée par Archimède; car il examinoit une ligne donnée, au lieu que *vous déterminez l'espace contenu dans une qui n'est pas encore donnée.* Je ne crois pas qu'il soit possible de trouver généralement la converse de ma règle pour les tangentes, &c." (*Œuvres de Descartes*, ed. Cousin,' tome VIII. p. 105, Paris 1824). Huyghens was the discoverer of evolutes (Scholium, p. 221): Tschirnhausen of caustics (Chasles *Aperçu historique* p. 110, 1875).

Maclaurin (*Geometria Organica*, sect. III. pp. 94 &c., London, 1720) propounded the theory of pedal and negative pedal curves. Notice, as a converse of Art. 38, that a conic may be regarded as the envelope of the arm YZ of a right angle inscribed in its auxiliary circle, whose other arm VZ passes through a focus H. Hence by projection, if two sides of a triangle inscribed in a given conic pass through fixed points C and H respectively, the envelope of the third side is a conic having double contact with the former on the line CH.

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[*] This corollary was suggested by Mr. R. R. Webb, Fellow of St. John's College.

We have seen that NEWTON proved the general tangent-property of conics, which was eventually presented in a new form, as a property of "double" or "anharmonic" ratio, by Steiner and Chasles (pp. 262, 295. Cf. Ex. 726, note). Chasles' second proof of Ex. 724 (*Quetelet's Correspondance &c.* v. 289), viz. by reciprocation with respect to a parabola, is interesting as an application of reciprocation to *metric* properties. See also Bobillier in *Gergonne's Annales* XVIII. 185; Poncelet, *Propriétés Projectives* II. 431 (1866); Booth, *New Geometrical Methods* vol. I. chap. 29.

The *Principle of Duality* was first fully brought out by Poncelet's method of reciprocal polars (Scholium E, p. 290). For some controversies on the discovery of the principle see his *Propriétés Projectives* II. 351—396. It has since been illustrated by the coordinate methods of Möbius, Plücker, Booth, &c. See also Chasles' *Aperçu historique* pp. 572—694, 1875; Townsend's *Modern Geometry* chap. 23. Figures which correspond according to the law of duality have been called by Chasles (p. 587) *Correlative* figures. They may also be called *Dual* figures. Any two dual figures are such that to every point in either corresponds a straight line in the other, and to every range in either a homographic pencil in the other, as is the case, for example, with reciprocal figures.

MINOR DIRECTRICES.

PROPOSITION VII.

158. *With either focus and directrix of an ellipse as origin and base line the major auxiliary circle reverses into a similar ellipse having for its minor auxiliary circle the reverse of the original ellipse.*

(i). For in Art. 4 it is evident that the major auxiliary circle of the ellipse reverses into an ellipse having the circle about O for its minor auxiliary circle; and by comparing the segments of the latus rectum of the obverse with the segments of the major axis of the reverse we see that $\frac{CB^2}{CA} \div CB$ in the one is equal to $\frac{CB}{CA}$ in the other. The two ellipses are therefore *similar*.

(ii). Let the annexed diagram represent the reverse figure, and let CA_0 be the major semiaxis of the obverse ellipse, and F the point on the minor axis of the reverse corresponding to C. Then F and its polar (the base line) are said to be a *Minor Focus* and *Directrix* of the reverse.

It is evident by parallels that

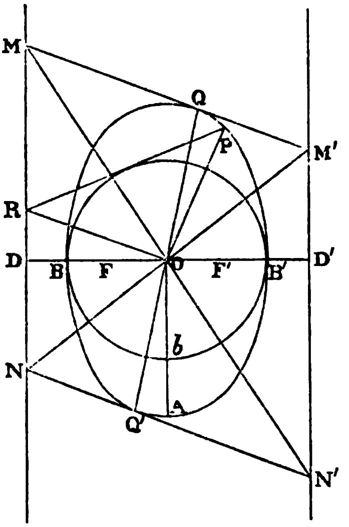
$$OF : OB : OD = CS : CA_0 : CX,$$

or

$$OF = \frac{CS}{CA_0} OB = e . OB,$$

where e is the common eccentricity of the two ellipses.

(iii). In the case of the Hyperbola, if $\pi - 2\chi$ and χ be the angles of the obverse and reverse hyperbolas respectively, and F as before the point corresponding to the centre C of the obverse, it may be shewn in like manner that F now lies on the *major* axis* of the reverse at a distance equal to $\frac{OA}{\sin \chi}$ from its centre O.



In the Rectangular Hyperbola the "major" and minor foci and directrices are coincident. In the Circle the minor foci coincide with the centre and the minor directrices are at infinity.

(iv). The properties of the minor directrices may also be arrived at by *Reciprocation*. It is easily seen that the reciprocal of an ellipse with respect to its major auxiliary circle is a similar ellipse having that circle for its minor auxiliary circle, &c.

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* Notice, in justification of the term *minor* axis in the general hyperbola, that the *square* of this axis, being negative, is always less than that of the transverse axis.

Notice, in illustration of this remarkable identity of results arrived at by reciprocation and reversion, that *if an ellipse and a hyperbola have the same axes, each is its own reciprocal with respect to the other.* [Ex. 326.

Corollary 1.

From S in the obverse draw SY to meet the major auxiliary circle and draw $S\infty$ at right angles to SY; then $Y\infty$ touches the conic (Art. 38). Hence, in the reverse figure, if OP be any radius of the conic and OR be drawn at right angles to it to meet the minor directrix, the envelope of PR is the minor auxiliary circle.

Corollary 2.

Every chord drawn through S to the major auxiliary circle of the obverse has its pole on the base line and makes equal angles with the tangents to the circle at its extremities. Hence any two parallel tangents QM, $Q'N$ drawn to the reverse conic and terminated by the minor directrix subtend equal angles at O.

SCHOLIUM B.

The theory of Minor Directrices* is due to BOOTH (*New Geometrical Methods* vol. I. 269), who investigated their properties by the method of reciprocation. In some cases he makes use of a double reciprocation, first reciprocating a figure A into B and then B into A'. But since figures reciprocal to the same figure are homographic with one another, it should be possible to derive the properties of A' directly from those of A. Take for example the property that *if a fixed straight line and the tangents from any point P upon it to an ellipse about O as centre meet one of its minor directrices in Q and T, T' respectively, then* $\tan\frac{1}{2}TOQ.\tan\frac{1}{2}T'OQ$ *is constant.* Regard the conic as the reverse of a circle, as in Prop. VII, and let qtt' be the points *at infinity* corresponding to QTT'. Then the angles TOQ, $T'OQ$ are equal to tSq, $t'Sq$ or tpq, $t'pq$ respectively; and it remains only to prove for the circle that $\tan\frac{1}{2}tpq\,.\tan\frac{1}{2}t'pq$ is constant. Compare the longer method of double reciprocation by which Booth establishes the proposition (loc. cit. p. 276). Similar remarks apply to his double reciprocation of umbilical quadrics (p. 208).

Taking O and ω as origins, let it be required to reverse three given points PQR in space into given points pqr respectively

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* They have also been called *secondary* directrices (*Proc. Royal Irish Academy* III. 503).

(Scholium, p. 328), each pair of points Pp, &c. being supposed to lie in a plane through the origins. Each pair Pp determine a point (OP, ωp) on the O-director and a point (Op, ωP) on the ω-director; and thus the three pairs Pp, Qq, Rr completely determine the two director planes. This construction is equivalent to reversing *five* given points $O\omega PQR$ into five given points $\omega Opqr$. Cf. Chasles, *Aperçu &c.* p. 754. If one director-plane be now removed to infinity, every two corresponding lines through the origins become parallel, as *in plano* (Art. 142).

From a given point O on an ellipsoid draw triads of chords OA, OB, OC at right angles, and let the *fixed* point of concourse of the planes ABC (cf. p. 324, note) and its polar plane be called the Frégier point and plane of O. Then, if O be taken as origin and its Frégier plane as director, the ellipsoid reverses into a sphere. If O be not on the surface, we may reverse the ellipsoid into a spheroid about ω as *centre*, and thus shew that the envelope of the planes ABC (Booth I. 97) is a quadric of revolution having O for a focus. [Art. 144.

159. *Examples of Reciprocation.*

We shall now give some illustrations of the method of applying the principles established above. The following theorems will be seen to be reciprocal:

If two vertices of a triangle slide on fixed straight lines whilst the sides pass each through a fixed point, the locus of the third vertex is a conic passing through the fixed points on the adjacent sides. [p. 264.	If two sides of a triangle pass through fixed points whilst the vertices slide each on a fixed straight line, the third side envelopes a conic touching the lines on which its extremities slide.
If two sides of a triangle inscribed in a conic pass each through a fixed point, the envelope of the third side is a conic having double contact with the former on the join of the fixed points.*	If two vertices of a triangle circumscribed to a conic slide each on a fixed line, the third describes a conic having double contact with the former upon the tangents to it from the intersection of the fixed lines.
The diameter of a quadrilateral is the centre-locus of all conics inscribed therein. [Ex. 832.	Given four points on a conic, the polar of a fixed point passes through a fixed point conjugate to the former with respect to every conic through the four given points. [p. 278.
The envelope of the polar of a given point with respect to a system of confocal conics is a parabola touching their axes, &c. [Ex. 822.	Given four points on a conic, the locus of the pole of a fixed straight line is a conic, &c. [Art. 125.

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* This is easily proved by projecting the conic and one of the fixed points into a circle and its centre (Art. 38).

The six centres of similitude of three arbitrary circles lie by threes on four straight lines.	If three conics have two common tangents (or a common focus), their six chords of intersection pass by threes through the same four points.
The locus of the centre of a circle which touches two given circles is a conic having their centres for foci.	The polar of the centre of a circle touching two given circles with respect to either of them envelopes another circle.
The centre of any one of the eight circles which touch three given circles may be determined as a point of concourse of two conics, each of which has for foci the centres of two of the given circles.*	The polar of the centre of a circle touching three given circles with respect to any one of the three may be determined as a common tangent of two other circles. The centre itself may then be determined from its polar.
A variable chord drawn through a fixed point O to a conic subtends a pencil in involution at any point on the curve. [Art. 120.	Parallel tangents to a conic (or tangents from points on a given straight line) determine an involution on any fixed tangent.
The product of the focal perpendiculars upon any tangent to a conic is constant.	The square of the distance of any point on a conic from a fixed origin varies as the product of its distances from two fixed right lines.

For if the polars of any two points P and F be taken with respect to *a circle whose centre is* O, and if Pf be a perpendicular to the polar of F and Fp a perpendicular to the polar of P, it is easily seen (Art. 153 § I.) that $OF.Pf = OP.Fp$.† If therefore F and F' be the foci of a conic and P any point on the curve, it follows that $OF.Pf.OF'.Pf' = OP^2.Fp.F'p'$, or OP^2 varies as the product of the perpendiculars from P to the reciprocals of F and F'.

160. *Angles. Confocal Conics.*

We shall next give some examples of the reciprocation of angles (Art. 155), and of confocal conics.

At any point on a circle the tangent is at right angles to the radius.	Any point on a conic and the intersection of the tangent thereat with the directrix subtend a right angle at the focus.
The polar of any point with respect to a circle is at right angles to the diameter through the point.	The pole of any straight line with respect to a conic and the point of concourse of the line with the directrix subtend a right angle at the focus.

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* See Salmon's *Conic Sections* Art. 317.

† For an independent proof see Macdowell's *Exercises in Euclid &c.* Art. 256.

Orthogonal tangents to a conic intersect on a concentric circle.	A chord of a conic which subtends a right angle at any fixed point envelopes a conic having that point and its polar for a focus and directrix.
Confocal conics intersect everywhere at right angles, and the tangents to two confocals from any point, taken alternately, include equal angles. [Art. 50.	Every common tangent of two circles subtends a right angle, and the opposite intercepts on any transversal subtend equal angles, at either limiting point.
If XX' and YY' be two pairs of collinear points on two circles, the tangents at XX' intersect those at YY' in four points lying on a third coaxal circle.*	If tangents be drawn from any point to two confocal conics, the four joins of the alternate points of contact touch a third confocal.
Given three pairs of lines conjugate with respect to a circle, every conic through the four positions of its centre is a rectangular hyperbola. [Ex. 812.	Given a focus O of a conic and three pairs of points conjugate with respect to it, there are four positions of the O-directrix, and the orthocycle of every conic touching the four passes through O.
Tangents being drawn in a given direction to a system of confocal conics, their points of contact lie on a rectangular hyperbola.	The tangents to the circles of a coaxal system at their points of concourse with a given transversal through either limiting point O envelope a conic, whose orthocycle passes through O.

161. *The Parabola.*

a. The reciprocal of a parabola with respect to any point O is a conic through O, and if O be the focus of the parabola the reciprocal is a circle through O, and conversely. Hence the following theorems are reciprocal:

If AOB be a right angle inscribed in a circle the hypotenuse AB passes through the centre.	The locus of the vertex of a right angle circumscribed to a parabola is the directrix.
The locus of the vertex of a right angle circumscribed to a parabola is the directrix.	A chord of any given conic which subtends a right angle at a fixed point O on the curve passes through a fixed point F on the normal at O.

Thus by a double reciprocation we deduce Frégier's theorem (Art. 144 § *b*) from a property of the circle. Notice that the *Frégier-point F* corresponds to the *directrix* of the parabola, the normal at O to the point at infinity on the parabola, the further extremity of the normal at O to the tangent at the vertex of the parabola, and that the tangent at O is parallel to the directrix.

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* See Townsend's *Modern Geometry* I. 264.

b. A triad of points ABC reciprocate with respect to their orthocentre into a triad of lines parallel to BC, CA, AB, and having the same orthocentre. Also a parabola reciprocates from any point O on its directrix into a rectangular hyperbola through O. Hence the following are reciprocal theorems:

The orthocentre of any triangle circumscribed to a parabola lies on the directrix.	The orthocentre of any triangle inscribed in a rectangular hyperbola lies on the curve.

The former is a special case of Brianchon's theorem (p. 290), the latter of Pascal's (p. 175).

c. If the reciprocals of three points ABC with respect to O be the lines abc, then to any point A' on BC corresponds the line through bc making an angle equal to AOA' with a. Hence—

The perpendiculars of any triangle circumscribed to a parabola meet on the directrix.	If from a fixed point O on a conic there be drawn any three chords OA, OB, OC and the three lines at right angles to them, and if the latter meet BC, CA, AB in A', B', C' respectively, then A', B', C' lie on a straight line passing through the Frégier point of O.

162. *The Minor Directrices.*

a. Let the tangent at Q to a conic meet the minor directrices in M and M', and let Fp and $F'p'$ be perpendiculars to this tangent and QT and QT' perpendiculars to the minor directrices. Then the angle QOM is equal to QOM' (Art. 158 Cor. 2), and therefore

$$OM : OM' = QM : QM' = QT : QT' = Fp : F'p',$$

$\dfrac{QT}{Fp}$ being to $\dfrac{QT'}{F'p'}$ in a *constant* ratio (Prop. III.) which is evidently a ratio of equality when Q is taken on either axis.

b. It may be seen that at any point Q on the conic,

$$QT.QT' : OQ^2 = OB^2 : e^2.OA^2,$$

since if Q be a point such that $\dfrac{QT.QT'}{OQ^2}$ is constant the locus of Q is a conic (Art. 159), and the above proportion requires that this conic should meet the axes in the same points as former. It then follows that

$$Fp.F'p' : OQ^2 = OY^2 : CA^2,$$

if OY be the central perpendicular to MM'.

c. If C be the pole of MM' with respect to the minor auxiliary circle, so that OM is at right angles to FC and OM' to $F'C$, and $FC+F'C=2OB$ (Art. 158 § iv), then, with the help of similar triangles, it may be shewn that, if X be the intersection of MM' with the major axis,

$$FC : OM = OF : OX = F'C : OM';$$

and therefore $DM+D'M'=2OX=e(OM+OM')$, D and D' being the points in which the minor directrices meet the minor axis.

d. The following are examples of reversion, or the same results may be obtained by reciprocation.

Any chord of a circle is at right angles to the diameter through its pole.	Any chord of a conic and the line joining the minor focus F to the pole of the chord meet the F-directrix in points which subtend a right angle at the centre.
If AB be fixed points on a circle and C any other point upon it, the angle ACB has one of two constant and supplementary values.	The arms of any angle ACB inscribed in given segment of a conic intercept on the minor directrices lengths which subtend constant and supplementary angles at the centre.

In like manner it may be shewn that the two pairs of opposite sides of a quadrilateral inscribed in a conic make intercepts on either minor directrix which subtend supplementary angles at the centre.

e. The major auxiliary circles of a system of conics having a focus S and its directrix in common may be reversed into concentric conics having the same minor directrices (Prop. VII.). Hence, the circles being coaxal and having S for a limiting point:*

The opposite intercepts made by any two circles on any transversal subtend equal angles at either limiting point. [Art. 160.	The opposite intercepts made by any two concentric conics having the same minor directrices upon any transversal subtend equal angles at the centre.

f. Each focus and directrix of a Rectangular Hyperbola being at the same time a minor focus and directrix (Art. 158 § iii), we obtain in this case the following property, corresponding to that of the ellipse in Art. 158 Cor. 1:

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* See Macdowell's *Exercises in Euclid &c.* Art. 251.

If $PD'DP'$ be a chord of a rectangular hyperbola touching its auxiliary circle and meeting the directrices in D and D', *the diameters through P and D (or P' and D') are at right angles.* Hence it follows also that PD and $P'D'$ subtend equal angles at the centre.

INVERSION.

163. We shall conclude with a slight sketch of the method of Inversion.

If O be a fixed point and P a variable point in a given plane, and if a point p be taken on OP or its complement such that $OP.Op$ is equal to a constant c^2, then p is said to be an *inverse* of P with respect to the pole O, and the locus of p is said to be the inverse of the locus of P. It is evident that *a straight line through the pole is its own inverse.*

To curves intersecting in points PQR &c. correspond curves intersecting at the inverse points pqr &c.; and therefore to curves having contact of any order at P correspond inverse curves having contact of the same order at the inverse point p.

164. *The inverses of any two curves intersect at the same angles as the original curves.*

For if PQ be any two points on a curve and pq the inverse points, then $OP.Op = OQ.Oq$, and therefore the angles OPQ and Oqp are equal. Hence, supposing P and Q to coalesce, the *tangents* at the inverse points P and p are equally inclined (on opposite sides) to the radius vector OPp. It then follows that the tangents to any two curves at a common point P are inclined at the same angles as the tangents to the inverse curves at p.

165. *The inverse of a straight line not passing through the pole is a circle through the pole, and conversely; and the inverse of a circle not passing through the pole is a circle.*

(i). To a given line draw a perpendicular OX from the pole of inversion, take any point D on the given line, and let the point x be inverse to X and d to D. Then since $Ox.OX = Od.OD$, and since OXD is a right angle, therefore Odx is a right angle, or the locus of d is a circle having its diameter through O at right angles to the given line.

Conversely *the inverse of a circle through O is a straight line at right angles to its diameter through O.*

It is further evident that the *tangent* to any curve at P inverts into a circle through O touching the inverse curve at the inverse point p; unless P coincides with O, in which case the tangent at that point is also an *asymptote* to the inverse curve.

(ii). The inverse of a circle not passing through O is a circle. For if OPQ be drawn to meet the given circle in P and Q, and if the point p be inverse to P, and q to Q, then since $OP.OQ$ is constant and $OP.Op$ is likewise constant; therefore Op varies as OQ, or the locus of p is similar to the locus of Q, which is a circle.

(iii). Notice that by making the constant of inversion equal to $OP.OQ$ we may invert the given circle into *itself*. If QQ' be a common tangent to two circles and M its middle point, then with M as pole and MQ^2 as the constant of inversion each of the circles inverts into itself. Again, if O be the centre and c the radius of a circle orthogonal to a set of coaxal circles, then with O as pole and c^2 as the constant of inversion the whole system inverts into itself.

166. *The nine-point circle of any triangle touches the inscribed and escribed circles.*

Let ABC be a triangle, I the inscribed circle touching BC in Q, and E the escribed circle opposite to A touching BC in Q'. Bisect BC, CA in M and M' respectively,[*] draw AP perpendicular BC, and let the nine-point circle N meet AP in D, which will be the further extremity of its diameter through M. Then, with M as pole and MQ^2 (equal to MQ'^2) as the constant of inversion, I inverts into itself, E into itself, and N into a *straight line* at right angles to MD.

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[*] See the lithographed figure No. 5. On the above proof see p. 191, note.

This straight line meets BC in a point R such that $MP.MR = MQ^2$, that is to say, in the point of concourse of BC with the line joining the centres of I and E;* and it makes an angle with BC equal to MDP, or $MM'P$, or $B \sim C$, and may therefore be shewn to coincide with the second tangent from R to I and E.

And since the inverse of N thus touches the inverse of I and the inverse of E, therefore N touches I and E, which are their own inverses; that is to say, it touches the inscribed circle and each of the escribed circles of the triangle ABC, as was to be proved.

To determine the points of contact, let RS be the second tangent from R to I (or E), produce MS to meet the circle again in S', then S' (the inverse of S) is the required point in which N touches I (or E).

167. *The Cardioid.*

The inverse of a parabola with respect to its focus is a cardioid having its cusp at the origin. Hence the following are inverse theorems:

The sum of the reciprocals of the segments of any focal chord of a parabola is constant.	The length of any cuspidal chord of a cardioid is constant.
Every focal chord of a parabola is divided harmonically by the focus and the directrix.	The locus of the middle points of the cuspidal chords of a cardioid is a circle through the cusp.
The tangents to a parabola at the extremities of a focal chord which makes an angle α with the axis are inclined at angles $\frac{\alpha}{2}$ and $\frac{\pi}{2} - \frac{\alpha}{2}$ to the chord.	The tangents to a cardioid at the extremities of a cuspidal chord inclined at an angle α to the axis make angles equal to $\frac{\alpha}{2}$ and $\frac{\pi}{2} - \frac{\alpha}{2}$ with the chord. [Art. 164.

Hence *the tangents to a cardioid at the extremities of any cuspidal chord intersect at right angles;* and it may now be shewn that the locus of their intersection is the circle concentric with the bisector of all cuspidal chords and of thrice its radius.

The tangents to a parabola at the extremities of any focal chord intersect at right angles on the directrix.	The two circles through the cusp which touch a cardioid each at one extremity of any cuspidal chord meet at right angles on a fixed circle through the cusp.

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* See Macdowell's *Exercises in Euclid &c.* Art. 86.

From a fixed point O draw OY to any point Y on a given straight line; then the envelope of the line through Y at right angles to OY is a parabola.	From a fixed point O on a given circle draw any chord Oy; then the envelope of the circle on Oy as diameter is a cardioid having its cusp at O.
The intersections of any three tangents to a parabola lie on a circle through the focus.	Any three circles touching a cardioid and passing through its cusp meet in three other points lying on a straight line.
If $ABCA'B'C'$ be any six points on a parabola, the intersections of $(AB, A'B')$, $(BC, B'C')$, $(CA, C'A')$ lie in a straight line.	If $abca'b'c'$ be any six points on a cardioid whose cusp is at O, the intersections of the three pairs of circles $(Oab, Oa'b')$, $(Obc, Ob'c')$, $Oca, Oc'a')$ lie on one circle.
If $ABCA'B'C'$ be any six tangents to a parabola, the joins of $(AB, A'B')$, $(BC, B'C')$, $(CA, C'A')$ meet in a point.	If $abca'b'c'$ be any six circles touching a cardioid and passing through its cusp, the three circles through the intersections of $(ab, a'b')$, $(bc, b'c')$, $(ca, c'a')$ are coaxal.

168. *Circles of Curvature.*

The osculating circle at any point P on a curve inverts from any point O into the circle (Art. 165) which osculates the inverse curve at the inverse point p. [Art. 163.

But if the former circle passes through O it inverts into a straight line, and p becomes a point of zero curvature, or of *inflexion*. Hence the following are inverse theorems:

Three points can be found on an ellipse whose osculating circles meet at a given point on the curve, and these three points lie on a circle through O.	The inverse of an ellipse with respect to any point upon it is a curve having three points of inflexion, which lie in a straight line.

It may be added that a parabola inverts from its vertex as pole into a *cissoid;* a central conic from either focus into a *limaçon*, and from its centre into an *oval of Cassini* or, if an equilateral hyperbola, into a *lemniscate of Bernoulli;* a conic from any point upon it into a *circular cubic* having a node at the pole; and a conic from any other point in its plane into a *trinodal quartic* having its nodes at the focoids and the pole.

SCHOLIUM C.

For the principle of Inversion Chasles (*Rapport* pp. 140—2) refers to Ptolemy, and to Quetelet (1827); and for a general statement of the method to Bellavitis (1836). In 1843—4 it was propounded afresh by Ingram and Stubbs (*Transactions of the Dublin*

Philosophical Society vol. I. 58, 145, 159; *Philosophical Magazine* XXIII. 338, XXV. 208). It has been applied by Dr. Hirst to attractions (*Phil. Mag.* 1858), and embodied by Peaucellier in his linkages. Cf. also *Camb. and Dublin Math. Journal* VIII. 47; *Oxf. Camb. Dubl. Messenger of Mathematics* III. 228; Booth's *New Geometrical Methods* vol. I. chap. 30; Salmon's *Higher Plane Curves*, Arts. 348 &c.; *Proc. London Math. Soc.* V. 105, VII. 91. And for a complete exposition of the method as applied to the straight line and circle see Townsend's *Modern Geometry*, chaps. 9, 24. This method, unlike projection and reciprocation, enables us to deduce properties of the higher curves from those of a lower order, and is thus peculiarly effective as an instrument of discovery and research.

MISCELLANEOUS EXAMPLES.

[851. If a triangle is self-polar to a parabola (p. 281), the three lines joining the middle points of its sides touch the parabola, and conversely. [Ex. 715.

852. Two rectangular hyperbolas being such that the axes of the one are parallel to the asymptotes of the other, and the centre of each lies on the other; shew that any circle through the centre of either meets the other again in a conjugate triad with respect to the former.

853. If two angles of given magnitudes turn about their summits A and B as poles, then (1) if one pair of their arms remain constantly parallel, the other pair intersect at a constant angle and thus describe a *circle* C through the poles; and (2) if one pair of their arms intersect on a fixed straight line D as director, the other pair by their intersection describe in general a *conic* through the poles. The points at infinity on the conic correspond to the intersections of C and D; the axes of the conic are parallel to the positions which the parallel arms in case (1) assume when the arms describing the circle intersect at the extremities of the diameter at right angles to D; and if the director be any line in a system of parallels the axes of the conics described are parallel.

854. If the tangent at O to a rectangular hyperbola be met at right angles in P by a chord QR, the diameters

bisecting OQ and OR bisect the angles between the diameters to O and P.*

855. If AA', BB', CC' be the three pairs of summits of a quadrilateral circumscribed to a parabola whose focus is S, then

$$SA.SA' = SB.SB' = SC.SC'.$$

856. Reciprocate the theorem that the feet of the focal perpendiculars upon the tangents to a parabola are collinear.

857. Given two conics, find a conic with respect to which they are polar reciprocals.

858. The tangent to a circle at any point makes with any chord through the point an angle equal to the angle in the alternate segment. What does this proposition become by reciprocation with respect to any origin?

859. The problem, to inscribe in a given conic a $2n$-gon whose n pairs of opposite sides shall pass in any assigned order through n given points, is always indeterminate or impossible.†

860. If two circles be drawn meeting a conic in $OABC$ and $OA'B'C'$ respectively, every two of the vertices of the triangles ABC and $A'B'C'$ subtend at O an angle equal to that between the opposite sides. Conversely, if the vertices of two triangles inscribed in a conic be thus related to a point O, then O lies on the conic.

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* The nine-point circle (Art. 64 Cor. 4) of OQR passes through O.

† This question and its solution were suggested by Prof. TOWNSEND. Starting from an arbitrary point P on the conic as one vertex, draw n successive sides of the polygon through the n points taken in the assigned order, and the other n sides through the same points taken in the reverse order; and let the points on the conic thus arrived at be Q and Q' respectively. As P varies, the three systems of points P, Q, Q' are homographic (Art. 120 Cor. 2), and therefore also the two systems $P + Q$ and $Q' + P$. If Q and Q' once coincide, one pair of homologous points in the two homographic systems $P + Q$ and $Q' + P$ are interchangeable, and therefore every pair are interchangeable (Townsend's *Modern Geometry* II. § 860), or Q and Q' always coincide. If then Q and Q' coincide for any *one* position of P the solution is indeterminate, and if not it is impossible.

861. The circumscribed circles of any two triangles PQR and $P'Q'R'$ circumscribing a parabola meet in a point O (other than the focus) such that the angle subtended at O by any two of the vertices of the triangles is equal to that between the opposite sides.* Hence shew that the seven points $OPQRP'Q'R'$ lie on a conic.

862. If three conics have one point in common, their nine common chords which do not pass through it touch one conic.† Conversely, if three triangles circumscribe a conic, the three conics which circumscribe them by pairs have one point in common. [Ex. 861.

863. Any three parabolas, taken in pairs, have three triads of common tangents, whose nine intersections lie on a conic. What does this become by projection and reciprocation?

864. If ABC be the triangle formed by three tangents to a parabola whose focus is S, the inclination of BC to the axis is equal to the angle subtended by SA at the circumference of the circle. Hence shew that the square of the radius of the circle is equal to $\dfrac{Sa\,.\,Sb\,.\,Sc}{p}$, if p be the parameter and abc the points of contact of the tangents.

865. Two triangles ABC and $A'B'C'$ inscribed in a circle being such that every two of their vertices subtend at the circumference an angle equal to that between the opposite sides, shew that, if O be the centre and OS a given radius of the circle,

$$\angle SOA + SOB + SOC = SOA' + SOB' + SOC',$$

and conversely. Hence deduce that, if ABC be a variable triangle circumscribed to a given parabola whose focus is S, and inscribed in a fixed circle whose centre is O, the sum of the angles SOA, SOB, SOC, measured in the same sense of rotation is constant; and the sum of the angles which BC, CA, AB make with the axis is constant. Shew also that the product $SA.SB.SC$, or $Sa.Sb.Sc$ (Ex. 864), is constant, and that the product of the focal perpendiculars upon the three tangents is constant.

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* If O be a point on the arc SP (p. 56) and r the point of contact of PQ, the angle ROS is equal to $RPQ + SPr$, and therefore to SrQ. In like manner $R'OS$ is equal to $Sr'Q'$; and therefore ROR' is equal to the angle $(PQ.\ P'Q')$.

† Picquet, *Etude géométrique des Systèmes Ponctuels &c.* pp. 27, 51 (Paris 1872).

866. If a triangle ABC inscribed in a given ellipse envelopes a fixed parabola, the sum of the eccentric angles of its vertices is constant;* and the circles ABC pass through a fixed point on the ellipse and have a common radical axis.

867. A variable conic through four given points $ABCD$ meets a fixed conic through D at the vertices of a variable triangle, which envelopes a fixed conic inscribed in ABC. State the reciprocal theorem.

868. The tangents to an ellipse from any point on a minor directrix intercept on the major axis a length which varies as the central distance of the point.

869. Any chord PQ of an ellipse and the tangents at P and Q meet the minor directrices in pairs of points RR', MM' and NN' respectively such that, if O be the centre, the angle ROR' is equal to $\frac{1}{2}(MOM' + NON')$; and the central distances of the points $MM'NN'$ and their perpendicular distances from PQ are to one another severally in the same ratio.

870. Prove by reciprocation with respect to a point, that the sum of the reciprocals of the perpendiculars from any point O within a circle to the tangents from any point on the polar of O is constant. Also prove that the reciprocals of equal circles with respect to the same point have equal parameters, and the reciprocals of coaxal circles with respect to any point on the radical axis have equal minor axes.†

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* This theorem, which is due to Mr. R. Pendlebury, Fellow of St. John's College, follows from Ex. 865 by orthogonal projection.

† If OAB be drawn from the origin to meet a given circle in A and B, then $\frac{c^2}{OA}$ and $\frac{c^2}{OB}$ are equal to the perpendiculars from a focus of the reciprocal conic to a pair of parallel tangents. The length of the minor axis is therefore determined by the product $OA.OB$.

871. Deduce the property of the focus and directrix of a conic from the *Locus ad quatuor lineas*.*

872. If P and Q be opposite intersections of the common tangents to two conics, any two lines OA and OB which are conjugate with respect to both conics are harmonic conjugates with respect to the lines OP and OQ. Deduce this from Newton's property of the diameter of a quadrilateral (Art. 124), viz. by projecting OA or OB to infinity. Also state the reciprocal theorem.† [I. 19.

873. An ellipse being drawn through the centre O of a circle, shew that the lines from O to a pair of opposite intersections of their common tangents are equally inclined to the tangent to the ellipse at O. [I. 33.

874. Find the locus of the centre of a conic which cuts four given finite straight lines harmonically, or which passes through two given points and cuts two given finite straight lines harmonically. [I. 62.

875. If a given polygon be moved about in its plane so that two of its sides touch each a fixed circle, every side of the polygon touches a fixed circle. [I. 68.

876. Deduce from Ex. 738, that if a curve has one tetrad of foci (p. 311) lying on a circle it has three other such tetrads, and the four circles cut one another orthogonally. [II. 10.

877. If four circles be mutually orthogonal, their centres form an orthocentric tetrastigm, and one at least of the circles is imaginary. [II. 10.

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* First deduce the *Locus ad tres lineas* $\alpha\beta = k\gamma^2$, and let the two tangents be drawn from the focus S. These are represented by the Cartesian equation $x^2 + y^2 = 0$, and the perpendiculars to them from (x, y) are proportional to $x \pm \sqrt{(-1)}\,y$. Therefore $x^2 + y^2$ varies as γ^2, or the distance of (x, y) from S varies as its distance from the polar of S.

† Exx. 872 &c. are taken from the EDUCATIONAL TIMES *Reprint*, the volume and page of which are specified in each case. See also pp. 141, 336.

878. Find the locus of the pole of a common chord of two fixed conics with respect to a conic having double contact with each of them. [II. 46.

879. If two triangles circumscribed (or inscribed) to a conic be in perspective, every radiant through their centre of perspective meets their sides in three pairs of points in involution. Reciprocate this theorem, and point out its relation to Steiner's property of the directrix of a parabola inscribed in a triangle. [II. 50.

880. If two conics meet any transversal in pairs of points AB and $A'B'$ respectively, the foci of the involution determined by AA' and BB' (or AB' and $A'B$) lie on a third conic passing through the intersections of the former two. [II. 91.

881. Given that one focus of a conic to which a given triangle is self-conjugate lies on a given straight line, find the locus of its second focus, and deduce Ex. 715. [III. 33.

882. Given two points P and Q on a conic, find a third point O upon it such that OP and OQ may divide a given finite straight line in a given cross ratio. [III. 47.

883. Let abc be the middle points of the sides of a triangle, O the centre of its circumscribing circle and O' its orthocentre. Then if Oa, Ob, Oc be produced to ABC respectively so that $OA=2Oa$, $OB=2Ob$, $OC=2Oc$, the sides of the triangle ABC and of the original triangle touch one conic, which has their common nine-point circle for its major auxiliary circle and the points O and O' for foci. [III. 53.

884. Through four given points draw a conic such that the chord which it intercepts on a given line shall be of given length, or shall subtend a given angle at a given point. [III. 84.

885. If p, p' be variable points collinear with a fixed point A, and so situated that the segment pp' always subtends a right angle at another fixed point M, prove the following properties of *corresponding* loci of p and p'. Right lines equidistant from the middle point of AM correspond to similar conics passing

through A and cutting AM perpendicularly at M. These conics are similar ellipses, or parabolas, or similar hyperbolas according as the common distance of the primitive lines from the middle point of AM is greater than, equal to, or less than $\frac{1}{2}AM$. The circles which pass through A and M, taken in pairs, are corresponding loci, as also are the circles which pass through M and have their centres on AM. [III. 91.

886. The critical conic of a quadrilateral being defined as the circumscribed conic which projects into a circle when the quadrilateral is projected into a square, shew that, if AA', BB', CC' be the three pairs of summits of a quadrilateral, a conic can be found having double contact at points lying on AA' with the critical conic of $BB'CC'$, double contact at points on BB' with the critical conic of $CC'AA'$, and double contact at points on CC' with the critical conic of $AA'BB'$. [III. 92.

887. If a straight line meet the sides BC, CA, AB of a triangle in PQR respectively, and O be any point in the same plane, the tangents at O to the conics $OAPBQ$ and $OAPCR$ are harmonic conjugates with respect to OA and OP. [IV. 44.

888. Shew how to prove the principal properties of the lemniscate by inversion. [IV. 47.

889. Prove that the "characteristics" of a system of conics satisfying four conditions are unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. [IV. 56.

890. Given four straight lines in a plane, we may project one of them to infinity and the remaining three into the sides of an equilateral triangle. Is it possible to project two given triangles at once into equilateral triangles, or a conic and a triangle into a circle and an equilateral triangle? [IV. 88.

891. The envelope of the circles on a system of parallel chords of a conic as diameters is a conic having its foci at the extremities of the diameter conjugate to the chords. Find where any circle touches the envelope. [V. 40.

892. A variable conic being drawn touching three given conics, if the normals at the three points of contact cointersect investigate the locus of their point of concourse. [v. 50.

893. Four conics being drawn through the same four points so that their tangents thereat form four harmonic pencils, shew that if one conjugate pair of the conics be a circle and an equilateral hyperbola the other pair must have equal eccentricities. [v. 103.

894. Through a given point O on a hyperbola two chords are drawn each at right angles to an asymptote, and from a variable point P on the curve perpendiculars PM and PN are drawn to the two chords through O. Shew that MN passes through a fixed point F; find the locus of F for different positions of O on the hyperbola; and determine the hyperbola for which the locus reduces to a point. [vi. 45.

895. Three conics being described so that each of them passes through the same point O and through the extremities of two of the diagonals of the same complete quadrilateral, prove that the remaining three points of concourse of the conics lie upon their tangents at O. [vi. 54.

896. If $ABCD$ be four points on a conic, the intersections of AB and CD with any two tangents lie on a conic touching AC and BD. [vi. 56.

897. The axes of every conic circumscribed to a quadrangle, which is itself inscribed in a circle, are parallel to two fixed right lines, viz. the asymptotes of the equilateral hyperbola (Ex. 518) which is the centre-locus of all conics circumscribed to the quadrangle.* [vi. 88.

898. If P be any point on a circle, A and B fixed points on a diameter and equidistant from the centre, the envelope of a transversal which is cut harmonically by the circles

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* If ABC be the common self-polar triangle of all the circumscribed conics; $O\infty$ and $O\infty'$ the axes of any one of them, so that $O\infty\infty'$ *is a self polar triangle* with respect to it; the points $ABCO\infty\infty'$ lie on a conic, which in this case is the rectangular hyperbola $ABCO$.

described with A and B as centres and AP and BP respectively as radii is the conic which has A and B for foci and touches the circle. [VII. 34.

899. Draw the minimum chord of a given angle which can be cut in a given ratio by a given line. [VII. 41.

900. If Q and R be the foci of any ellipse inscribed in a triangle ABC, deduce from Ex. 322 that,

$$AQ.AR.BC+BQ.BR.CA+CQ.CR.AB=BC.CA.AB.$$

[VII. 43.

901. If Q be the intersection of the polars of any point P with respect to two given parallel conics (Ex. 792, note), the locus of the middle point of PQ is their radical axis. Hence shew that, if DEF be the feet of the perpendiculars of any triangle ABC, and AL, BM, CN be parallels to EF, FD, DE, meeting BC, CA, AB in the points LMN respectively, then the axis of perspective of the triangles ABC and DEF bisects each of the segments AL, BM, CN. [VII. 78.

902. Construct geometrically the four chords of contact with a given conic of the four inscribed conics which pass through three given points. [VII. 92.

903. Triads of parallels being drawn through the vertices ABC of a given triangle to meet the opposite sides in abc, shew that the envelope of the axis of perspective of the triangles ABC and abc is the maximum ellipse that can be inscribed in ABC. [VII. 94.

904. If from any point on a conic parallels be drawn to the diameters bisecting the sides of any inscribed triangle, the lines so drawn meet the corresponding sides of the triangle in three collinear points. Extend this theorem by projection, and also reciprocate it. [VIII. 44.

905. Prove by inversion, that the circles having for diameters three chords AB, AC, AD of a circle intersect again by pairs in three collinear points. [VIII. 48.

906. If through a pair of opposite intersections AA' of four fixed tangents to a given conic there be drawn a pair of lines conjugate to the conic, the locus of their point of concourse is a conic passing through AA' and through the points of contact of the four tangents. [VIII. 62.

907. If A be any point within or without a conic, B any point on its polar, CD a fixed straight line, BC and BD tangents cutting CD in C and D respectively; shew that the intersections of AD, BC and AC, BD lie on a fixed straight line, which meets CD on the polar of A. [VIII. 63.

908. If DP and DQ be a pair of tangents to a conic and ABC a self-polar triangle, any conic through $ABCD$ cuts PQ harmonically. Hence shew that the perpendicular DM to either axis bisects the angle PMQ. [VIII. 110.

909. Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatres points harmoniques toute conique inscrite dans le quadrilatère. [IX. 62, XII. 50.

910. La condition qu'une conique divise harmoniquement les trois diagonales d'un quadrilatère circonscrit à une autre conique, coincide avec la condition que la première conique soit circonscrite à un triangle conjugué à l'autre. [IX. 74.

911. The degree of the locus of the foci of a system of conics subject to four conditions is three times as great as that of the locus of their centres. [X. 63.

912. If A and B are fixed points with regard to a conic of which ACD is a variable chord, shew that the polar of A meets BC and BD in points E and F such that AB, DE, CF cointersect. [X. 81.

913. Given three points ABC and a straight line through each, shew how to cut the three lines by a fourth in points PQR such that the lengths AP, BQ, CR may be equal. [XI. 19.

914. If three equidistant lines parallel to an asymptote of a hyperbola meet the curve in ABC, prove by involution (or

otherwise) that any parallel to the other asymptote is divided harmonically by the sides of the triangle ABC and the curve. [XI. 20.

915. The six points which, in conjunction with any common transversal, divide harmonically the six sides of a tetrastigm, lie on a conic passing also through the three intersections of the opposite sides of the tetrastigm; and the three straight lines which join the six points in opposite pairs cointersect at the pole of the transversal with respect to the conic. [XI. 21.

916. Reciprocate the theorem, that if the orthocentre of a triangle inscribed in a parabola lies on the directrix, the circle to which the triangle is self-polar passes through the focus. [XI. 32.

917. Extend by projection and also reciprocate the following theorem. Given two parallel conics (Ex. 792, note) A and B, two circles can be drawn having double contact with A and B respectively and meeting their common chord in the same two points. [XI. 43.

918. If O and O' be the limiting points of a system of coaxal circles, and if with O and O' respectively as one focus two conics be described osculating any circle of the system at one and the same point, their corresponding directrices will coincide. [XI. 74.

919. Given three points ABC and a conic, the envelope of a chord PQ such that $A\{BPQC\}$ is harmonic is a conic touching AB and AC at points lying on the polar of A. [XI. 83.

920. Find the envelope of a transversal on which two given conics intercept segments having a common middle point M, and find the locus of M. [XI. 84.

921. Any tangent to a conic is divided in involution by three other tangents and the radiants to their intersections from either focus S. Prove that the double points of this involution, as the tangents vary, subtend a pencil in involution at S. [XI. 105.

922. PQ being a chord of a conic equally inclined to the axis with the tangent at P, a circle is drawn through PQ cutting the conic again in RS. Shew that the point on the circle harmonically conjugate to P with respect to R and S lies upon the chord of the conic supplemental to QP. What does this become by inversion? [XII. 90.

923. Any two parallel tangents to a conic meet the tangents from a given point O in points T and T' respectively such that $OT.OT'$ is constant. [XIII. 44.

924. Prove by inversion, that if three circles meet two and two in AA', BB', CC', and O be any point in their plane, the circles OAA', OBB', OCC' are coaxal. [XIV. 102.

925. Given in a conic two tetrastigms $PQRS$ and $pqrs$ whose corresponding chords pass by fours through the same three points, shew that a conic may be drawn touching Pp, Qq, Pr, Ss at $pqrs$ respectively. [XIV. 104.

926. Find the constant ratios which five fixed radiants in space determine on a variable transversal plane;* and deduce the anharmonic property of four radiants in one plane. [XV. 26.

927. Prove, generalise and reciprocate the theorem, that the bisectors of the angles between the two pairs of opposite sides of a trapezium inscribed in a circle are at right angles. [XV. 36.

928. The envelope of a transversal cut harmonically by two given similarly situated parabolas is a third parabola (Ex. 800). [XV. 86.

929. The tangents to a conic from a variable point on a fixed straight line L meet the tangent at a given point A in R and R'. Shew that the relation between AR and AR' is of the form $a.AR.AR'+b.AR+c.AR'+d=0$ (Ex. 777); and determine the positions of L in order that (1) the sum of the intercepts AR and AR', or (2) the sum of their reciprocals may be constant. [XVI. 59.

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* See the *Messenger of Mathematics* vol. v. 94 (1876).

930. Two sides of a triangle circumscribed to a given circle being fixed, the three lines joining its angles severally to the points of contact of the escribed circles with the opposite sides meet in a point, whose locus is a hyperbola having the fixed sides for asymptotes. [XVI. 62.

931. Given three fixed straight lines lmn and three fixed points LMN in a straight line, the lines from a current point on l to M and N meet m and n in four points, the conic through which and L envelopes a conic touching m and n at their intersections with l. [XVI. 98.

932. The first positive and negative pedals of an equilateral hyperbola are reciprocal polars with respect to it. [XVI. 106.

933. Given a point on one arm of a constant angle inscribed in a circle, find the envelope of the other arm. [XVI. 110.

934. Circles being described on the two halves of a diameter of a given circle as diameters, shew that the perpendicular radius of the given circle is trisected by the centre and circumference of a fourth circle touching the three; and deduce a new theorem by reciprocation. [XVII. 23.

935. Deduce from Ex. 785, that if BC be a chord of a circle and A its pole, the conic through ABC which touches the circle at a point D has its curvature at D twice as great as that of the circle. [XVII. 109.

936. An ellipse having double contact with a fixed ellipse E has one focus F fixed: shew that the other focus describes an ellipse confocal with E and passing through F. [XVIII. 70.

937. The area of the triangle formed by the polars of the middle points of the sides of a triangle with respect to any inscribed conic is equal to the area of the given triangle. [XVIII. 107.

938. From two fixed points on one of a series of confocal conics tangents are drawn to a variable conic of the series: if they meet the fixed conic again in QR and $Q'R'$, shew that the locus of the point $(QR, Q'R')$ is a conic. [XIX. 51.

939. The ratio of the product of the diameters of two circles to the square of one of their common tangents inverts into an equal ratio when the circles are inverted from any point as pole. Hence deduce Feuerbach's property* of the nine-point circle. [XIX. 54.

940. In a quadrilateral whose diagonals intersect at right angles shew how to inscribe a conic having their intersection for a focus. [XIX. 69.

941. A point C being taken on the diameter AB of a semicircle, semicircles are described on AC and BC as diameters. Also a series of circles are described, the first touching the three semicircles, and every n^{th} circle touching the $n-1^{\text{th}}$ and the semicircles on AB and AC. Prove that, as C varies, the loci of the centres of the several circles are ellipses having a common focus. [XIX. 88.

942. A line being drawn from the focus of a conic to meet the tangent at a constant angle, find where the locus of the point of concourse touches the conic. [XIX. 111.

943. The tangent at any point of a cardioid meets the curve again in two points PQ the tangents at which divide the double tangent AB harmonically; and the locus of the point of concourse of the tangents at PQ is a conic passing through AB and touching the cardioid at one real and two imaginary points. [XX. 34.

944. If a lamina moves in its own plane so that two given points of it describe each a fixed straight line, any other point of the lamina describes an ellipse. [XX. 89.

945. If ABC be three points on a parabola, $A'B'C'$ the intersections of the tangents thereat, and abc the centres of the circles BCA', CAB', ABC'; prove that the circle abc passes through the focus. [XXI. 72.

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* On this see Schröter in Neumann's *Mathematische Annalen* vol. VII. 517–530 (Leipzig 1874), where the property is cited from Feuerbach's *Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks* (Nürnberg 1822). The nine-point circle itself has been improperly called Feuerbach's by Baltzer and others.

946. The tangent to the evolute of a parabola where it cuts the parabola is also a normal to the evolute. [XXI. 79.

947. Find at what points on a conic the angle between the tangent and the chord drawn to a fixed point on the curve is greatest or least. [XXII. 29.

948. If OP and OQ be tangents to a conic, R the middle point of PQ, and O' the point harmonically conjugate to O with respect to the foci on the circle through O and the foci, shew that $OP.OQ=OR.OO'$; and deduce that if O and the foci be fixed the circles OPQ are coaxal.* [XXIII. 17.

949. If DD' be a fixed diameter of a conic and AB any two conjugate points in an involution on the tangent at D', then DA and DB meet the conic again upon a chord which passes through a fixed point. [XXIII. 55.

950. If AB be the base of a segment of a parabola and P any point on the curve, the locus of the orthocentre of APB is a line parallel to AB. Hence shew how to describe a parabolic segment of given base and height by points. [XXIII. 61.

951. A plane figure moves so that two fixed straight lines in it always pass through two fixed points: find the envelope of any straight line in the figure. [XXIII. 67.

952. One focal chord of a conic meets the tangents at the extremities of another in A and B. If straight lines ACD and BEF be drawn perpendicular to AB and meeting the curve in $CDEF$, prove that CE and DF meet AB at a point P on the directrix; that CF and DE, AF and DB, AE and BC meet on the polar of P; that the intercepts CD and EF subtend equal angles at the focus S; that $SA : SC : SD = SB : SE : SF$; that CF and DE meet AB in two points G and H, having properties like those of A and B; and that of the four intersections of the tangents from A and B two lie on the polar of P and two on the directrix. [XXIV. 21.

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* Compare Exx. 322, 337, 340, 380.

953. Given a conic inscribed in a parallelogram, if any tangent to the conic meets the sides opposite to an angle A in B and C, prove that the triangle ABC is of constant area. [XXIV. 51.

954. Six circles pass through twelve points on a conic in the order,

(*a*) $A_1A_2A_3A_4$, (*b*) $B_1B_2B_3B_4$, (*c*) $C_1C_2C_3C_4$,
(*d*) $A_1A_2B_3C_4$, (*e*) $B_1B_2C_3A_4$, (*f*) $C_1C_2A_3B_4$;

prove that the circles *abc* meet the circles *def* in six new points which lie on the circumference of another circle. [XXIV. 75.

955. Prove that there are eight chords of an ellipse normal to the curve at one extremity and to the central radius vector at the other. [XXIV. 83. XXV. 73.

956. If $ABCD$ be a quadrilateral inscribed in a conic, F and G the intersections of its opposite sides; prove that every conic through $ACFG$ has with the given conic a chord of intersection which passes through a fixed point, viz. the pole of BD. [XXIV. 93.

957. If PP' be points on equal circles whose centres are O and O' respectively, and if the lines OP', $O'P$ be parallel, find the envelope of the line bisecting both. [XXV. 53.

958. What is the condition that the conic $\alpha\beta = k\gamma^2$ may touch the conic $\alpha\beta = \gamma^2$ externally? [XXV. 88.

959. Given five points $ABCDE$ on a conic, shew that there is a sixth point on it the parallel through which to AB passes through the fourth point of concourse of the circle CDE with the conic. [XXVI. 17, 103.

960. If six lines taken in the orders 1231′2′3′, 123′1′2′3, 12′31′23′, 12′3′1′23 respectively form hexagons each inscribed in a conic, each pair of the conics have a common chord in the same straight line with a common chord of the opposite pair; and nine of the common chords are the sides and the joins of the opposite vertices of two triangles in perspective, one of which is inscribed in the other. [XXVI. 21.

961. If two points be such that the tangents to a parabola from the one are at right angles to the tangents from the other, the loci of the two points are in perspective. What does this become by projection and reciprocation? [XXVI. 94.

962. The joins of n points on a conic intersect again in three times as many points as there are combinations of n things taken four together, and of these intersections one third lie within and two thirds without the curve. [XXVI. 101.

963. If the three pairs of opposite summits of a quadrilateral be severally conjugate with respect to a conic, the joins of the poles of its diagonals cut the conic in a hexagon to which the sides of the quadrilateral are Pascal lines.

[XXVII. 105.

964. Shew that there are in general eight positions of a chord of an ellipse which meets the curve at given angles at its two extremities. [XXVIII. 63.

965. Three conics $S_1S_2S_3$ being such that the polar reciprocal of any one with respect to another is the third, a triangle ABC is inscribed in S_1 and circumscribed to S_2. Prove that the triangle determined by the points of contact is self-polar to S_1 and circumscribes S_3; and that the tangents to S_1 at ABC form a triangle self-polar to S_2 inscribed in S_3.

[XXVIII. 97.

966. If S be the focus of a conic inscribed in a triangle ABC, and any tangent meet the focal chords perpendicular to SA, SB, SC in PQR respectively, prove that AP, BQ, CR meet in a point. [XXVIII. 99.

967. A variable circle being drawn through two given points, through one of which pass two given lines; find the envelope of the chord joining the other points in which the circle meets them. [XXIX. 24.

968. If four conics $SABC$ have one focus and a tangent D in common, and if a common tangent to each of the pairs (SA), (SB), (SC) meet a directrix of ABC respectively upon the tangent D; the common tangents of BC, CA, AB meet at three points in a straight line. [XXIX. 43.

969. If three conics touch one another and have a common focus, the common tangent of any two cuts the directrix of the third in three points on one straight line. [XXIX. 69.

970. Prove the following pairs of reciprocal properties of a system of two conics:

a. When two conics are such that two of their four common points subtend harmonically the angle determined by the tangents at either of the remaining two, they subtend harmonically that determined by those at the other also.

b. When two conics are such that two of their four common tangents divide harmonically the segment determined by the points of contact of either of the remaining two, they divide harmonically that determined by those of the other also.

c. The associated conic, envelope of the system of lines divided harmonically by the two original conics, breaks up in the former case into the point-pair determined by the eight tangents to them at their four common points.

d. The associated conic, locus of the system of points subtended harmonically by the two original conics, breaks up in the latter case into the line-pair determined by their eight points of contact with their four common tangents. [XXIX. 88.

971. If the sides of a variable triangle pass severally through three fixed points in a straight line, whilst one vertex moves on a straight line and a second describes a given curve; prove that the locus of the third vertex is homographic with the given curve. [XXIX. 96.

972. The triangles whose vertices are two triads of points on a conic intersect in nine points, such that the join of any two not on the same side is a Pascal line of the six vertices. [XXX. 25.

973. If a system of conics having a common focus envelop a given curve, and have their eccentricities proportional to the focal distances of the poles of their directrices with respect to a circle about the common focus as centre, the locus of the poles is a parallel of the reciprocal of the given curve with respect to the circle. [XXX. 93.

974. Prove and also reciprocate the following theorem. If a circle A touches a circle B internally at P, and if the tangent to A at any point Q meets B in R_1 and R_2, then $\angle R_1PQ = \angle R_2PQ$. [XXXI. 65.

975. Two lines being drawn through any point P on an ellipse to meet the major axis in A, B and the minor axis in a, b respectively; shew that if $PA = PB = \frac{1}{2}$ minor axis, the intersection of Ab and Ba is the Frégier-point of P. [XXXII. 48.

976. If a circle and a rectangular hyperbola intersect in four points, the line joining their centres is bisected by the centroid of the four points. [XXXII. 48.

977. If PQ is a chord normal at P to an ellipse, and N the intersection of the normal at Q with the tangent at P, then PN is to the projection of the semi-diameter CP upon it as the square of PQ to the square of the conjugate semi-diameter. [XXXII. 58.

978. The focal radii to the points in which a fixed tangent to an ellipse meets a variable pair of conjugate diameters intersect on a fixed circle. [XIII. 33. XXXII. 81.

979. If four parallel chords of an ellipse $\alpha\beta\gamma\delta$ be met by a straight line in $abcd$ respectively, shew that

$$a^2.bc.cd.db + \gamma^2.da.ab.bd = \beta^2.cd.da.ac + \delta^2.ab.bc.ca.$$

[XXXIII. 27.

980. If P be a current point on a given segment AB, the ellipses of given eccentricities described with AP and BP respectively as foci intersect upon a fixed ellipse whose foci are A and B. [XXXIII. 52.

981. If Pp and Qq be chords of a parabola parallel to the tangents at q and p respectively, and Oo the poles of PQ and pq; shew that $\Delta OPQ = 27 \Delta opq$, and that, if pq be parallel to a fixed line, the locus of the intersection of PQ with the tangents at p and q is a similarly situated parabola. [XXXIII. 58.

982. The locus of the foot of the perpendicular from any point on a given diameter of a conic to its polar is a rectangular hyperbola. [XXXIII. 76.

983. Any focal chord being drawn to a hyperbola, the circle on the portion of it intercepted by the tangents at the vertices as diameter touches the hyperbola. [XXXIII. 110.

984. The envelope of the axes of a conic which touches four fixed tangents to a circle is a parabola.[*]

985. Shew how to solve the problems of the two mean proportionals and of the trisection of an angle by the intersections of a circle and a parabola.[†]

986. The axes of a conic, any chord and the normals at its extremities touch one parabola. Deduce a construction for the centre of curvature at any point.[‡]

987. If the tangent and normal at any point of a conic meet the major and minor axes in TG and tg respectively, the radius of curvature at the point subtends a right angle at (Gt, gT). To what does this reduce in the case of the parabola?

988. If from any point of a conic a line equal to the radius of curvature be drawn normally outwards, the circle upon it as diameter is orthogonal to the orthocycle. What does this become in the case of the parabola? and what in the case of the rectangular hyperbola?[||]

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[*] One triangle ABC is self-polar to every conic inscribed in the given quadrilateral; and the axes of any one of them produced to infinity determine a second such triangle $O \infty \infty'$ with respect to it. The conic inscribed in ABC and $O \infty \infty'$ is a fixed parabola, whose directrix is easily seen to be the diameter of the quadrilateral, since the orthocentre of ABC is the centre of the circle.

[†] See Descartes *Geometria* lib. III p. 91 (ed. Schooten, 1659).

[‡] Exx. 862, 986–992, &c. are to be found, with or without solutions, in Steiner's posthumous work *Vorlesungen über synthetische Geometrie*, Theil. II pp. 80, 206–212, 222–3, 242 (ed. 2, 1876). On his theorem Ex. 998 see *Nouvelles Annales* XIV. 103 (1855); Housel *Introd. à la Géométrie Supérieure* p. 231 (Paris 1865).

[||] In the rectangular hyperbola the diameter of curvature at any point is equal to the normal chord, as Mr. Wolstenholme thus proves. Take on the curve three points ABC and their orthocentre O; then OA produced to meet the circle through ABC again is bisected by BC, and its halves, when ABC coalesce, become the normal chord and the diameter of curvature at A.

989. If $a\alpha$, $b\beta$, $c\gamma$ be three couples in an involution, shew that

$$\{a\alpha bc\}.\{\beta bca\}.\{\gamma cab\} = \{a\alpha\beta\gamma\}.\{b\beta\gamma\alpha\}.\{c\gamma\alpha\beta\} = -1,$$

where $\{a\alpha bc\}$ denotes the cross ratio $\frac{ab}{\alpha b} : \frac{ac}{\alpha c}$. Also if $abcde$ be any five lines in a plane, shew that

$$\{e.abcd\}.\{c.abde\}.\{d.abec\} = 1.$$

990. If OA, OB, OC, OD are concurrent normals to a conic, the parabola which touches the tangents at $ABCD$ touches the axes and the polar of O, has the diameter through O for its directrix,* and is the polar reciprocal of the rectangular hyperbola through $ABCD$ (Art. 114 Cor. 1) with respect to the original conic. Determine the focus of the parabola.† [Ex. 379.

991. The tangents from any point of an ellipse to its auxiliary circles are equal to the real and imaginary semi-axes of the confocal hyperbola through the point.

992. If $abcd$ and $\alpha\beta\gamma\delta$ be two tetrads of points on a conic, the joins of $(ab, \alpha\beta)$ and $(cd, \gamma\delta)$; $(ac, \alpha\gamma)$ and $(bd, \beta\delta)$; $(ad, \alpha\delta)$ and $(bc, \beta\gamma)$ meet in a point. And if $a\alpha$, $b\beta$, $c\gamma$ be concurrent chords of a conic, then

$$\{a\alpha b\gamma\} + \{\beta bc\alpha\} + \{\gamma ca\beta\} = 1.$$

993. If $a\alpha$, $b\beta$, $c\gamma$ are the foci of three conics inscribed in the same quadrilateral, then

$$ac.\alpha c : a\gamma.\alpha\gamma = bc.\beta c : b\gamma.\beta\gamma.$$

994. If a conic A circumscribes a conic B harmonically, then B is harmonically inscribed to A; the reciprocals of A are

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* As a second proof that O lies on the directrix, Mr. Pendlebury remarks that the normals are also normals at points $abcd$ to the reciprocal of the conic with respect to O, so that the parabola has for its reciprocal the *rectangular hyperbola Oabcd*, and therefore subtends a right angle at O (Art. 155 Cor. 2). Thus also we see that the normals meet the original conic again in points $A'B'C'D'$ which lie on a rectangular hyperbola through O; since the reciprocals of $ABCD$ touch a parabola having for directrix the diameter through O to the reciprocal of the original conic, and (by symmetry) the reciprocals of $A'B'C'D'$ touch a parabola having the same directrix.

† The parabola is the same for all confocal conics.

harmonically inscribed to the reciprocals of B; and the centre of perspective of any triangle inscribed in A and its reciprocal with respect to B lies on A.*

·995. A conic is harmonically inscribed to every circle orthogonal to its orthocycle; and a rectangular hyperbola harmonically circumscribed to a circle passes through its centre.

996. The asymptotes of a conic are conjugate lines with respect to any parabola harmonically inscribed to it.

997. Given that a focus of one conic is a point O on the orthocycle of another, if one of the conics be harmonically inscribed to the other, it touches the polar of O with respect to the latter.

998. Describe the conic with respect to which five given pairs of lines are conjugate; and the conics which pass through 4, 3, 2, 1 or 0 given points and are harmonically circumscribed to 1, 2, 3, 4 or 5 given conics.

999. The orthocycles of the conics which touch two given lines SA and SB at given points A and B, including the circle on AB as diameter and the point-circle at S, are coaxal.†

1000. The number of conics touching five given conics is 3264.‡

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* On Exx. 994–8 see Picquet's *Etude géométrique* &c. pp. 58, 91, 108, 181–8; Prof. H. J. S. Smith *On some Geometrical Constructions* (Proceedings of the London Mathematical Society vol. II. 85–100). One conic is said to be harmonically inscribed or circumscribed to another when it is inscribed or circumscribed to a triangle self-polar with respect to the latter.

† Gaskin *Geometrical Construction &c.* p. 31. Hence (Ex. 577 note) the theorem, lately pointed out by Mr. R. W. Genese and Mr. Torry, that *the directrix of a conic is a common chord of the conic, its orthocycle and a point-circle at the focus.* Notice that every straight line through a focoid, as being an asymptote or self-conjugate diameter of a circle (pp. 142, 309), is *at right angles to itself.*

‡ See Salmon's *Conic Sections* (end); Halphen *Proc. London Math. Soc.* vol. IX. 149 and X. 87; and the original memoir by Chasles, *Détermination du nombre des sections coniques qui doivent toucher cinque courbes données d'ordre quelconque, ou satisfaire à diverses autres conditions* (Comptes Rendus LVIII. 225, 1864).

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